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# A HEAT CONDUCTION PROBLEM AND CERTAIN PRODUCTS OF THE MULTIVARIABLE *H*-FUNCTION WITH TWO GENERAL CLASSES OF POLYNOMIALS

By

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## ABSTRACT

Special functions and their applications are becoming increasingly important to boundary value problems in partial differential equations of engineering and physics. So far as its technical applications is concerned, space research and nuclear reactor also give rise to several problems of the applications of special functions. The aim of this paper is to discuss the applications of certain products of the multivariable $H$-function with two general classes of polynomials in obtaining a solution of the partial differential equation

$$\frac{\partial\theta}{\partial t} = L\,\frac{\partial^2\theta}{\partial x^2}$$

related to a (simple but typical) problem of heat conduction in a rod with one end at zero degree and the other end insulated. The results so derived may be found useful for computing different values of $f(x)$ for several special functions (in one and more arguments).

**1. INTRODUCTION** As an example of the applications of certain products of the multivariable $H$-function with two general classes of polynomials in applied mathematics, we shall consider the problem of obtaining a solution of heat conduction problem in one dimensional rod with constant thermal properties having no source, with one end at zero degree and the other end insulated, such that no solutions of heat equation develop exponentially in time.

Now, we formulate a simple (but typical) problem of heat flow to find a formula for the temperature $\theta(x, t)$ in one dimensional rod $0 \le x \le F$ must satisfy the heat equation

$$\frac{\partial\theta}{\partial t} = L\,\frac{\partial^2\theta}{\partial x^2}, \quad t \geq 0 \qquad \ldots(1)$$

and the initial conditions

$$\theta\,(0, t) = 0 \qquad \ldots(2)$$

$$\frac{\partial\theta}{\partial x}\,(E, t) = 0 \qquad \ldots(3)$$

$$\theta\,(x, t) \text{ is finite when } t \to \infty \qquad \ldots(4)$$

$$\theta\,(x, 0) = f(x) \qquad \ldots(5)$$

Here we shall consider

$$f(x) = \sin\left(\frac{\pi x}{F}\right)^{\gamma-1} S_n^m\left[y\left(\sin\frac{\pi x}{F}\right)^h\right] S_{n'}^{m'}\left[y'\left(\sin\frac{\pi x}{F}\right)^{h'}\right]$$
$$H\left\{z_1\left(\sin\frac{\pi x}{F}\right)^{2h_1}, \ldots, z_r\left(\sin\frac{\pi x}{F}\right)^{2h_r}\right\}, \qquad \ldots(6)$$

where the multivariable $H$-function is the $H$-function of several complex variables introduced by Srivastava and Panda ([8]; see also [6], p.251) and for the general class of polynomials ([5]; see also [7]).

## 2. THE MAIN INTEGRAL

$$\int_0^F\left(\sin\frac{\pi x}{F}\right)^{\gamma-1}\sin\,(2v+1)\left(\frac{\pi x}{2F}\right)S_n^m\left[y\left(\sin\frac{\pi x}{F}\right)^h\right]$$
$$S_{n'}^{m'}\left[y'\left(\sin\frac{\pi x}{F}\right)^{h'}\right]H\left(z_1\left(\sin\frac{\pi x}{F}\right)^{2h_1}, \ldots, z_r\left(\sin\frac{\pi x}{F}\right)^{2h_r}\right)dx$$
$$= F\,2^{1-\gamma}\sin\,(2v+1)\left(\frac{\pi}{4}\right)\sum_{s=0}^{(n/m)}\frac{(-n)_{ms}}{s!}A_{n,s}\,(2^h\,y)^s$$
$$\sum_{s'=0}^{(n'/m')}\frac{(-n')_{m's'}}{s'!}A_{n',s'}\,[2^{h'}\,y']^{s'}\;H^{0,\lambda+1\,:\,(u',\,v');\ldots;(u^{(r)},v^{(r)})}_{A+1,\,C+2:\,[B',D'];\ldots;[B^{(R)},\,D^{(r)}]}$$
$$\left(\begin{matrix}[1-\gamma-hs-h's':2h_1;\ldots;2h_r],..[(a):\theta':\ldots;\theta^{(r)}]: \\ [(c):\psi';\ldots;\psi^{(r)}],[(1-2\gamma-2hs-2h's-2v)/4:h_1;\ldots;h_r],[(3-2\gamma-2hs-2h's'+2v)/4;h_1;\ldots;h_r]:\end{matrix}\right.$$
$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)});\phi^{(\rho)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\;z_1 4^{-h_1};\ldots;z_r 4^{-h_r}\right) \ldots(7)$$

where $\mathrm{Re}\left(\gamma + \sum_{i=1}^{r} h_i d_j^i/\delta_j^i\right) > 0,\ h_i > 0,\ h > 0,\ h' > 0,$

$$|\arg (z_i)| < T_i \frac{\pi}{2}, T_i > 0, 1 \le i \le r,$$

$$1 \le j \le u^{(i)},$$

and $m$ and $m'$ are arbitrary integers and the coeffients $A_{n,s}$ and $A_{n',s'}$ $(n, n', s, s' \ge 0)$ are constants, real or complex.

The integral in (7) can be established by making use of the definition of the general class of polynomials ([8]; see also [6]) and a result recently obtained by Sharma ([3], p.372).

## 3. SOLUTION OF THE PROBLEM

The solution of the problem will be found to be

$$\theta(x, t) = 2^{2-\gamma-hs-h's'} \sum_{v=0}^{\infty} \sin (2v+1)\left(\frac{\pi}{4}\right) e^{-(2v+1)(\pi/2F) Lt}$$

$$\sin\left(v+\frac{1}{2}\right)\left(\frac{\pi x}{F}\right) \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} A_{n,s} y^s \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!} A_{n',s'} y^{s'}$$

$$H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[l-\gamma-hs-h's':2h_1;\ldots;2h_r]\\ [(c):\psi';\ldots;\psi^{(r)}],\end{matrix}\right.$$

$$[(a):\theta';\ldots;\theta^{(r)}]:$$
$$[(1-2\gamma-2hs-2h's'-2v)/4:h_1;\ldots;h_r],[(3+2v-2\gamma-2hs-2h's')/4;h_1;\ldots;h_r];$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix} z_1 4^{-h_1}, \ldots, z_r 4^{-h_r}\right) \quad \ldots(8)$$

where $h > 0, h' > 0, h_i > 0, \mathrm{Re}\left(\gamma + \sum_{i=1}^{r} h_i d_j^i/\delta_j^i\right) > 0,$

$$|\arg (z_i)| < T_i \frac{\pi}{2}, T_i > 0, 1 \le i \le r, 1 \le j \le u^{(i)},$$

and $m$ and $m'$ are arbitrary integers and the coefficients $A_{n,s}$ $(n, s \ge 0)$ and $A_{n',s'}$ $(n', s' \ge 0)$ are arbitrary constants, real or complex.

The general solution of the problem expressed in section 1 by setting $\theta = X(x)\ T(t)$ can be easily established as ([4], p.75)

$$\theta(x, t) = \sum_{v=0}^{\infty} B_v \sin\left(\left(v+\frac{1}{2}\right)\frac{\pi x}{F}\right) e^{\{-(2v+1)\pi/2F\} Lt} \quad \ldots(9)$$

If $t = 0$, then by virtue of (5) we have

$$\left(\sin\frac{\pi x}{F}\right)^{\gamma-1} S_n^m\left[y\left(\sin\frac{\pi x}{F}\right)^h\right] S_{n'}^{m'}\left[y'\left(\sin\frac{\pi x}{F}\right)^{h'}\right]$$

$$H\left(z_1\left(\sin\frac{\pi x}{F}\right)^{2h_1}, \ldots, z_r\left(\sin\frac{\pi x}{F}\right)^{2h_r}\right) = \sum_{v=0}^{\infty} B_v \sin\left(v+\frac{1}{2}\right)\left(\frac{\pi x}{F}\right) \quad \ldots(10)$$

Multiplying both the sides of (10) by $\sin (w + \frac{1}{2}) \frac{\pi x}{F})$, integrating with respect to $x$ from 0 to $F$ and using (7) along with the orthogonality property of sines, we get

$$B_w = 2^{2-\gamma-hs-h's'} \sin (2w+1)\frac{\pi}{4}) \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} A_{n,s}\, y^s$$

$$\sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!} A_{n',s'}\, y'^{s'} H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];\ldots;[B^{(r)},D^{(r)}]}$$

$$\left(\begin{matrix} [1-\gamma-hs-h's' : 2h_1;\ldots;2h_r], \ldots [(a):\theta';\ldots;\theta^{(r)}]: \\ [(c):\psi';\ldots;\psi^{(r)}], [(1-2\gamma-2hs-2h's')/4 : h_1;\ldots;h_r]; [(3-2\gamma-2hs-2h's'+2w)/4 : h_1;\ldots;h_r)]: \end{matrix}\right.$$

$$\left.\begin{matrix} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix}\; z_1 4^{-h_1}, \ldots, z_r 4^{-h_r}\right), \quad \ldots(11).$$

Putting the values of $B_w$ from (11) in (9), we get the related solution.

## 4. APPLICATIONS AND PARTICULAR CASES

**(A)** By applying our results given in (7) and (8) to the case of Hermite polynomials ([9], p.106, eq. (5.54) and [7], p.158) by setting

$$S_n^2(x) \to x^{n/2} H_n\left(\frac{1}{2\sqrt{x}}\right)$$

in which case $m = 2$, $A_{n,s} = (-1)^s$ and also letting $m' = 2$. $A_{n',s'} = (-1)^{s'}$. We have the following interesting consequences of the main results :

**(A.1)**

$$\int_0^F \sin\left(\frac{\pi x}{F}\right)^{\gamma-1} (\sin (2v+1)\frac{\pi x}{2F})\left[y\left(\sin\frac{\pi x}{F}\right)^h\right]^{n/2} H_n\left[\frac{1}{2\sqrt{y}\,(\sin\frac{\pi x}{F})^h}\right]$$

$$\left[ y' (\sin \frac{\pi x}{F})^{h'} \right]^{n'/2} H_{n'}\left[ \frac{1}{2\sqrt{y' (\sin \frac{\pi x}{F})^{h'}}} \right]$$

$$H\left( z_1(\sin \frac{\pi x}{F})^{h_1}, \ldots, z_r (\sin \frac{\pi x}{F})^{h_r} \right) dx$$

$$= F.2^{1-\gamma-n-n'-hs-h's'} \sin ((2v+1)\frac{\pi}{4}) \sum_{s=0}^{n/2} \frac{(-n)_{2s}}{s!} (-\frac{y}{4})^s$$

$$\sum_{s'=0}^{(n'/2)} \frac{(-n')_{2s'}}{s'!} (\frac{-y'}{4})^{s'} H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];\ldots;[B^{(r)},D^{(r)}]}$$

$$\left( \begin{matrix} (1-\gamma-hs-h's');2h_1;\ldots;2h_r], [(a):\theta';\ldots;\theta^{(r)}]: \\ [(c):\psi';\ldots;\psi^{(r)}], [(1-2\gamma-2hs-2h's'-2v)/4;h_1;\ldots;h_r][(3-2\gamma-2hs-2h's'+2v)/4:h_1;\ldots;h_r]: \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(\rho)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} z_1 4^{-h_1},\ldots, z_r 4^{-h_r} \right) \quad \ldots(12)$$

which holds true under the same conditions needed for (7).

**(A.2)**

$$\theta(x,t) = 2^{2-\gamma+n+n'-hs-h's'} \sum_{v=0}^{\infty} \sin (2v+1)\frac{\pi}{4}) e^{-[(2v+1)\pi/2F] Lt}.$$

$$\sin (v+\frac{1}{2})\frac{\pi x}{F}) \sum_{s=0}^{[n/2]} \frac{(-n)_{2s}}{s!} (-\frac{y}{4})^s \sum_{s'=0}^{(-n'/2)} \frac{(-n')_{2s'}}{s'!} (-\frac{y'}{4})^{s'}$$

$$H^{0,\lambda+1:(u',v');\ldots;(u^{(r)}, v^{(r)});}_{A+1,C+2:[B',D'];\ldots;[B^{(r)}.D^{(r)}];} \left( \begin{matrix} [1-\gamma-hs-h's':2h_1;\ldots;2h_r] \\ [(c):\psi;\ldots;\psi^{(r)}] \end{matrix} \right.$$

$$[(a):\theta';\ldots;\theta^{(r)}]:$$

$$[(1-2\gamma-2hs-2h's'-2v)/4:h_1;\ldots;h_r], [(3+2v-2\gamma-2hs-2h's')/4;h_1;\ldots;h_r]:$$

$$\left. \begin{matrix} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} z_1 4^{-h_1},\ldots, z_r 4^{-h} \right) \quad \ldots(13)$$

valid under the same conditions as obtainable from (8).

**(B)** For the Laguerre polynomials ([9], p.101, Eq. 15, 1.6) and [7], p.159), setting $S_n^1(x) \to L_n^{(\alpha')}(x)$ in which case

$m = 1, A_{n,s} \binom{n+\alpha'}{n} (\frac{1}{(\alpha'+1)_k})$ and also taking $m' = 1$.

$A_{n',s'} \binom{n'+\alpha''}{n'} (\frac{1}{(\alpha''+1)_{k'}})$ the results in (7) and (8) reduce to the following formulae :

**(B.1)**

$$\int_0^F \left(\sin \frac{\pi x}{F}\right)^{\gamma - 1} \sin\left((2v+1)\frac{\pi x}{2F}\right) L_n^{\alpha'}\left[ y \left( \sin \frac{\pi x}{F} \right)^h \right] L_{n'}^{\alpha''}\left[ y' \left( \sin \frac{\pi x}{F} \right)^{h'} \right]$$

$$H\left( z_1 \left( \sin \frac{\pi x}{F} \right)^{2h_1}, \ldots, z_r \left( \sin \frac{\pi x}{F} \right)^{2h_r} \right) dx$$

$$= F . 2^{1 - \gamma - hs - h's'} \sin\left((2v+1)\frac{\pi}{4}\right) \sum_{s=0}^{(n)} \binom{n+\alpha'}{n-s} \frac{(-y)^s}{s!} \sum_{s'=0}^{(n')} \binom{n'+\alpha''}{n'-s'} \frac{(-y')^{s'}}{s'!}$$

$$H^{0, \lambda+1 : (u', v');\ldots;(u^{(r)}, v^{(r)})}_{A+1, C+2 : [B', d'];\ldots;[B^{(r)}, D^{(r)}]} \left( \begin{matrix} 1 - \gamma - hs - h's' : 2h_1;\ldots;2h_r] \\ [(c) : \psi';\ldots;\psi^{(r)}] \end{matrix} \right.$$

$$[(a) : \theta' ;\ldots; \theta^{(r)}]:$$

$$[(1 - 2\gamma - 2hs - 2h's' - 2v)/4) : h_1;\ldots;h_r], [(3 - 2\gamma - 2hs - 2h's' + 2v)/4 : h_1,\ldots,h_r]:$$

$$\left. \begin{matrix} [(b') : \phi'] ;\ldots; [(b^{(r)}) : \phi^{(r)}]; \\ [(d') : \delta'] ;\ldots; [(d^{(r)}); \delta^{(r)}]; \end{matrix} \; z_1 4^{-h_1} ;\ldots; z_r 4^{-h_r} \right) \qquad \ldots(14)$$

valid under the same conditions as obtainable from (7).

**(B.2)**

$$\theta(x, t) = 2^{2 - \gamma + n + n' - hs - h's'} \sum_{v=0}^{\infty} \sin\left(2v+1\right)\frac{\pi}{4}) \; e^{-[(2v+1)\pi/2F]\, Lt}$$

$$\sin\left(v+\frac{1}{2}\right)\frac{\pi x}{F}) \sum_{s=0}^{n} \frac{(-y)^s}{s!} \binom{n+\alpha'}{n-s} \sum_{s'=0}^{n'} \frac{(-y')^{s'}}{s'!} \binom{n'+\alpha''}{n'-s'}$$

$$H^{0, \lambda+1 : (u', v');\ldots;(u^{(r)}, v^{(r)}):}_{A+1, C+2 : [B', D'];\ldots;[B^{(r)}, D^{(r)}]:} \left( \begin{matrix} [1 - \gamma - hs - h's' : 2h_1 ;\ldots; 2h_r] \\ [(c) : \psi' ;\ldots; \psi^{(r)}], \end{matrix} \right.$$

$$[(a) : \theta' ;\ldots; \theta^{(r)}]:$$

$$[(1 - 2\gamma - 2hs - 2h's' - 2v)/4 : h_1;\ldots;h_r], [(3 + 2v - 2\gamma - 2hs - 2h's')/4 : h_1;\ldots;h_r]:$$

$$\left. \begin{matrix} [(b') : \phi'] ;\ldots; [(b^{(r)}) : \phi^{(r)}]: \\ [(d') . \delta'] ;\ldots; [(d^{(r)}) . \delta^{(r)}]: \end{matrix} \; z_1 4^{-h_1}, \ldots, z_1 4^{-h_r} \right) \qquad \ldots(15)$$

holds true under the same conditions as those required for (8).

**(C)** for the Jacobi polynomials ([9], p.68, Eq. (4.3.2) and [7], p.159), setting $S_n^1(x) \to P_n^{(\alpha', \beta')}(1-2x)$ and $S_n^1(x) \to p_{n'}^{(\alpha'', \beta'')}(1-2x)$ in which case $m = 1$, $A_{n,s} = \binom{n+\alpha'}{n} \frac{(\alpha' + \beta' + n + 1)_s}{(\alpha' + 1)_s}$ and also let $m' = 1$.

$$A_{n', s'} = \binom{n'+\alpha''}{n'} \frac{(\alpha'' + \beta'' + n' + 1)_{s'}}{(\alpha'' + 1)_{s'}}.$$

From equations (7) and (8), we obtain the following results.

**(C.1)**

$$\int_0^F (\sin \frac{\pi x}{F})^{\gamma - 1} \sin (2v + 1) \frac{\pi x}{2F}) \, p_n^{(\alpha', \beta')} \left[ 1 - 2y \left( \sin \frac{\pi x}{F} \right)^h \right]$$

$$P_{n'}^{(\alpha'', \beta'')} \left[ 1 - 2y' \left( \sin \frac{\pi x}{F} \right)^{h'} \right] H \left(z_1 (\sin \frac{\pi x}{F})^{2h_1}, \ldots, z_r (\sin \frac{\pi x}{F})^{2h_r} \right) dx$$

$$= F.2^{1 - \gamma - hs - h's'} \sin (2v + 1) \frac{\pi}{4}) \sum_{s=0}^{n} (-y)^s \binom{n + \alpha'}{n - s} \binom{\alpha' + \beta' + n + s}{s}$$

$$\sum_{s'=0}^{(n')} (-y')^{s'} \binom{n' + \alpha''}{n' - s'} \binom{\alpha'' + \beta'' + n' + s'}{s'}$$

$$H^{0, \lambda + 1 : (u', v');\ldots;(u^{(r)}, v^{(r)})}_{A + 1, C + 2 : [B', D'];\ldots;[B^{(r)}, D^{(r)}]} \left( \begin{matrix} [1 - \gamma - hs - h's' : 2h_1;\ldots;2h_r], \\ [(c) : \psi';\ldots;\psi^{(r)}], \end{matrix} \right.$$

$$[(a) : \theta';\ldots;\theta^{(r)}]:$$
$$[(1 - 2\gamma - 2hs - 2h's' - 2v)/4v : h_1;\ldots;h_r] \; [(3 - 2\gamma - 2hs - 2h's' + 2v)/4 : h_1;\ldots;h_r]:$$

$$\left. \begin{matrix} [(b') ; \phi'] ;\ldots; [(b^{(r)}) : \phi^{(r)}]; \\ (d') : \delta'];\ldots; [(d^{(r)}); \delta^{(r)}]; \end{matrix} \; z_1 4^{-h_1} ;\ldots; z_r 4^{-h_4} \right), \qquad \ldots(16)$$

valid under the same conditions needed for (7).

**(C.2)**

$$\theta (x, t) = 2^{2 - \gamma - hs - h's'} \sum_{v=0}^{\infty} \sin (2v + 1) \frac{\pi}{4}) \, e^{[-(2v + 1) \pi/2F] Lt}$$

$$\sin (v + \frac{1}{2}) \frac{\pi x}{F}) \sum_{s=0}^{n} (-y)^s \binom{n + \alpha'}{n - s} \binom{\alpha' + \beta' + n + s}{s}$$

$$\sum_{s'=0}^{n'} (-y')^{s'} \binom{n' + \alpha''}{n' - s'} \binom{\alpha'' + \beta'' + n' + s'}{s'}$$

$$H^{0, \lambda + 1 : (u', v');\ldots;(u^{(r)}, v^{(r)})}_{A + 1, C + 2 : [B', D'];\ldots;[B^{(r)} : D^{(r)}]} \left( \begin{matrix} [1 - \gamma - hs - h's' : 2h_1;\ldots;2h_r], \\ [(c) : \psi';\ldots;\psi^{(r)}], \end{matrix} \right.$$

$$[(a) : \theta';\ldots;\theta^{(r)}]:$$
$$[(1 - 2\gamma - 2hs - 2h's' - 2v)/4 : h_1;\ldots;h_r], [(3 - 2\gamma - 2hs - 2h's' + 2v)/4 : h_1;\ldots;h_r]:$$

$$\left. \begin{matrix} [(b') : \phi'] ;\ldots; [(b^{(r)}) : \phi^{(r)}]: \\ [(d') : \delta'] ;\ldots; [(d^{(r)}) : \delta^{(r)}]; \end{matrix} \; z_1 4^{-h_1};\ldots; z_r 4^{-h} \right) \qquad \ldots(17)$$

valid under the same conditions as obtainable from (8).

**(D)** Taking $\lambda = A$, $u^{(i)} = 1$, $v^{(i)} = B^{(i)}$ and replacing $D^{(i)}$ by $D^{(i)} + 1$, $\forall\ i = 1, \ldots, v$ in (7) and (8), we get

**(D.1)**

$$\int_0^F (\sin \frac{\pi x}{F})^{\gamma - 1} \sin (2v + 1) \frac{\pi x}{2F}) \, S_n^m \left[ y (\sin \frac{\pi x}{F})^h \right] S_{n'}^{m'} \left[ y' \left( \sin \frac{\pi x}{F} \right)^{h'} \right]$$

$$F^{A : B' ;\ldots; B^{(r)}}_{C : D' ;\ldots; D^{(r)}} \left( \begin{matrix} [1-(a) : \theta' ;\ldots; \theta^{(r)}] : [1-(b') : \phi'] ;\ldots; [1-(b^{(r)}) : \phi^{(r)}] ; \\ [1-(c) : \psi' ;\ldots; \psi^{(r)}] : [1-(d') : \delta'] ;\ldots; [1-(d^{(r)}) : \delta^{(r)}] ; \end{matrix} \right.$$

$$\left. -z_1 (\sin \frac{\pi x}{F})^{2h_1}, \ldots, -z_r (\sin \frac{\pi x}{F})^{2h_r} \right) dx$$

$$= F . 2^{1 - \gamma - hs - h's'} \sin ((2v + 1) \frac{\pi}{4}) \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} A_{n,s} \, y^s \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!} A_{n',s'} \, y^{s'}$$

$$\frac{\Gamma (\gamma + hs + h's')}{\Gamma [(3 + 2\gamma + 2hs + 2h's' + 2v)/4] \, \Gamma [(1 + 2\gamma + 2hs + 2h's' - 2v)/4]}$$

$$F^{A+1 : B' ;\ldots; B^{(r)}}_{C+2 : D' ;\ldots; D^{(r)}} \left( \begin{matrix} [\gamma + hs + h's' : 2h_1 ;\ldots; 2h_r] ; \\ [1-(c) : \psi' ;\ldots; \psi^{(r)}] \end{matrix} \right.$$

$$[1-(a) : \theta' ;\ldots; \theta^{(r)}] :$$

$$[(3 + 2\gamma + 2hs + 2h's' + 2v)/4 : h_1 ;\ldots; h_r], [(1 + 2\gamma + 2hs + 2h's' - 2v)/4 : h_1 ;\ldots; h_r] :$$

$$\left. \begin{matrix} [1-(b') : \phi'] ;\ldots; [1-(b^{(r)}) : \phi^{(r)}] ; \\ [1-(d') : \partial'] ;\ldots; [1-(d^{(r)}) ; \delta^{(r)}] ; \end{matrix} \; -z_1 \, 4^{-h_1}, \ldots, -z_r \, 4^{-h_r} \right), \qquad \ldots(18)$$

valid under the same conditions as required in (7).

**(D.2)**

$$\theta (x, t) = 2^{2 - \gamma - hs - ns'} \sum_{v=0}^{\infty} \sin (2v + 1) \frac{\pi}{4}) \, e^{-[(2v+1)\pi/2F] Lt}$$

$$\sin ((v + \frac{1}{2}) \frac{\pi x}{F}) \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} A_{n,s} \, y^s \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!} A_{n',s'} \, y'^{s'}$$

$$\frac{\Gamma (\gamma + hs + h's')}{\Gamma [(3 + 2\gamma + 2hs + 2h's' + 2v)/4] \, \Gamma [(1 + 2\gamma + 2hs + 2h's' - 2v)/4]}$$

$$F^{A+1 : B' ;\ldots; B^{(r)}}_{C+2 ; D' ;\ldots; D^{(r)}} \left( \begin{matrix} [\gamma + hs + h's' : 2h_1 ;\ldots; 2h_r] \\ [1-(c) : \psi' ;\ldots.; \psi^{(r)}] : \end{matrix} \right.$$

$$[1-(a) : \theta' ;\ldots: \theta^{(r)}] :$$

$$[(3 + 2\gamma + 2hs + 2h's' + 2v)/4 : h_1 ;\ldots; h_r], [(1 + 2\gamma + 2hs + 2h's' - 2v)/4 : h_1 ;\ldots; h_r] :$$

$$\left. \begin{matrix} [1-(b') : \phi'] ;\ldots; [1-(b^{(r)}) : \phi^{(r)}] ; \\ [1-(d') : \delta'] ;\ldots; [1-(d^{(r)}) : \delta^{(r)}] ; \end{matrix} \; -z_1 \, 4^{h_1}, \ldots, -z_r \, 4^{-h_r} \right) \qquad \ldots(19)$$

valid under the same conditions as obtainable from (8).

**(E)** Letting $n' \to 0$ the results in (12) through (19) follow as particular case of the results recently obtained by Sharma [3].

**(F)** Taking $n \to 0$ and $n' \to 0$, the results in (7) and (8) reduce to the results recently obtained by Chaurasia ([1], p.172, eq.(2.1) and p.173, eq.(3.1)).

The importance of our results lies in its manifold generality. In view of the generality of the *H*-function of several complex variables, on specializing the various parameters and variables in the *H*-function of several complex variables, we obtain, from our results, several integrals and solutions involving a remarkably wide variety of useful functions (or product of several such functions), which are expressible in terms of *E, F, G* and *H* functions of one and several variables.

Secondly, by suitably specializing the coefficients $A_{n,s}$ and $A_{n',s'}$ and making a free use of the special cases of $S_n^m$ [$x$] listed by Srivastava and Singh [7], our results can be reduced to a large number of integerals and solutions of the problem (and also its particular cases) involving a product of generalized Hermite polynomials, Jacobi polynomials and its various special cases, Laguerre polynomials, Bessel polynomials. Gould-Hopper polynomials, Brafman polynomials and their various combinations. Thus the results presented in this paper would at once yield a very large number of results involving a large variety of polynomials and various special functions occuring in the problems of mathematical analysis, applied mathematics and mathematical physics.

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## REFERENCES

[1] V.B.L. Chaurasia, Application of the multivariable H-function in heat conduction in a rod with one end at zero degree and the other end insulated *Jñānābha* **21** (1991), 171-176.

[2] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, New York, 1980.

[3] R.P. Sharma, *A Study of Integral Tansforms and the H-functions of Several Complex Variables with their Applications*, Ph.D. Thesis, Univ. of Rajasthan, 1994.

[4] A. Sommerfield, *Partial Differential Equations in Physics,* Academic Press, New York, 1949.

[5] H.M. Srivastava, A Contour integral involving Fox's *H*-function, *Indian J. Math.* **14** (1972), 1-6.

[6] H.M. Srivastava, K.C. Gupta and S.P. Goyal, *The H-functions of One and Two variables with Applications,* South Asian Publishers, New Delhi and Madras, 1982.

[7] H.M. Srivastava and N.P. Singh, The integration of certain products of the multivariable H-function with a general class of polynomials, *Rend. Circ. Mat. Palermo* (2) **32** (1983), 157-187.

[8] H.M. Srivastava and Rekha Panda, Some bilateral generating functions for a class of generalised hypergeometric polynomials, *J. Reine Angew. Math.* **283/284** (1976), 265-274.

[9] G. Szegö, *Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ.,* Vol. **23**, Fourth edition, Amer. Math. Soc. Providence, Rohde Island, 1975.

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# STABILITY OF TWO RIVLIN-ERICKSEN ELASTICO-VISCOUS SUPERPOSE FLUIDS

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## ABSTRACT

The stability of the plane interface separating two Rivilin-Ericksen elastico-viscous superposed fluids of uniform densities has been studied. The stability analysis has been carried out, for mathematical simplicity, for two highly viscous fluids of equal kinematic viscosities and equal kinematic viscoelasticities. It is found that the system is stable for stable configuration and unstable for unstable configuration. The behaviour of growth rates with respect to kinematic viscosity and kinematic viscoelasticity parameters are examined analytically.

**1. INTRODUCTION** The instability of the plane interface separating two fluids when one is accelerated towards the other or when one is superposed over the other has been studied by several authors, and Chandrasekhar [2] has given a detailed account of these investigations. Roberts [5] has extended the analysis to the case of two fluids of equal kinematic viscosities in presence of vertical magnetic field while Gerwin [3] has studied the case of compressible streaming fluids. The influence of viscosity on the stability of the plane interface separating two incompressible superposed fluids of uniform densities, when the whole system is immersed in a uniform horizontal magnetic field has been studied by Bhatia [1]. He has carried out the stability analysis for two fluids of equal kinematic viscosities and different uniform densities.

Sharma and Sharma [6] have studied the stability of the plane interface separating two viscoelastic (Oldroydian) superposed fluids of uniform densities in another study Sharma [7] has studied the instability of the plane interface between two Oldrydian viscoelastic superposed conducting fluids in the presence of a uniform magnetic field. With the growing importance of non-Newtonian fluids in modern technology and industries, the further investigations on such fluids are desirable. There are many non-Newtonian fluids that cannot be characterized by Oldroyd's [4] constitutive relations. Rivlin-Ericksen elastico- viscous fluid is one such fluid. It is this class of elastico-viscous fluids we are interested in, particularly to study the stability of the plane interface separating two imcompressible superposed Rivlin-Ericksen fluids of uniform densities. This aspect forms the subject matter of the present paper wherein we have carried out the stability analysis for two fluids of equal kinematic viscosities, equal kinematic viscoelasticities and different uniform densities.

**2. PERTURBATION EQUATIONS** Consider a static state, in which an incompressible Rivlin- Ericksen elastico-viscous fluid of variable density is arranged in horizontal strata and the pressure $p$ and density $\rho$ are functions of the vertical coordinate $z$ only. The character of the equilibrium of this initial static state is determined, as usual, by supposing that the system is slightly disturbed and then by following its further evolution.

Let $\vec{v}(u, v, w)$, $\delta\rho$ and $\delta p$ denote the perturbations in velocity (0,0,0), density $\rho$ and pressure $p$, respectively. Then the linearized perturbation equations relevant to the problem are

$$\rho \frac{\partial \vec{v}}{\partial t} = -\nabla \delta p + \vec{g}\delta\rho + \rho\left(\nu + \nu' \frac{\partial}{\partial t}\right) \nabla^2 \vec{v} + \left(\frac{\partial \mu}{dz} + \frac{\partial}{\partial t}\frac{\partial \mu'}{dz}\right)\left(\frac{\partial w}{\partial \bar{x}} + \frac{\partial \vec{v}}{\partial z}\right) \quad \ldots(1)$$

$$\nabla \cdot \vec{v} = 0 \quad \ldots(2)$$

$$\frac{\partial}{\partial t}(\delta\rho) = -wD\rho, \quad \ldots(3)$$

where $\nu \left(= \frac{\mu}{\rho}\right)$ and $\nu' \left(= \frac{\mu'}{\rho}\right)$ denote respectively the kinematic viscosity and the kinematic viscoelasticity of the fluid, $\vec{g} = (0, 0, -g)$ is the acceleration due to gravity $\bar{x} = (x, y, z)$ and $D = \frac{d}{dz}$. Equation (3) ensures that the density of every particle remains unchanged as we follow it with its motion.

Analyzing the disturbances into normal modes, we assume that the perturbed quantities have the space and time dependence of the form

$$f(z) \exp(ik_x x + ik_y y + nt), \quad \ldots(4)$$

where $k_x$, $k_y$ are the wave numbers along the $x$ and $y$ directions respectively, $k = (k_x^2 + k_y^2)^{1/2}$ is the resultant wave number, $n$ is the growth rate which is, in general, a complex constant and $f(x)$ is some function of $z$.

For perturbations of the form (4), equations (1)-(3) become

$$\rho n u = -ik_x\,\delta p + \rho(\nu + \nu' n)(D^2 - k^2)u + (D\mu + nd\mu')(ik_x w + Du) \qquad ...(5)$$

$$\rho n v = -ik_y\delta p + \rho(\nu + \nu' n)(D^2 - k^2)v + (D\mu + nd\mu')(ik_y w + Dv), \qquad ...(6)$$

$$\rho n w = -D\delta p - g\delta\rho + \rho(\nu + \nu' n)(D^2 - k^2)w + 2(D\mu + nd\mu')dw \qquad ...(7)$$

$$ik_x u + ik_y v + Dw = 0 \qquad ...(8)$$

and

$$n\delta\rho = -wD\rho \qquad ...(9)$$

Multiplying equations (5) and (6) by $-ik_x$ and $-ik_y$, respectively, adding the resultant equations and using equation (8) in it, we obtain

$$\rho n Dw = -k^2\delta p + \rho(\nu + \nu' n)(D^2 - k^2)Dw + (D\mu + nD\mu')(D^2 + k^2)w \quad ...(10)$$

Substituting for $\delta\rho$ from (9) in equation (7) and eliminating $\delta p$ between equations (7) and (10), we obtain the equation in $w$ :

$$n[D(\rho Dw) - k^2\rho w] - [D\{\rho(v + v'n)(D^2 - k^2)Dw\}$$

$$- k^2\rho(\nu + \nu' n)(D^2 - k^2)w] + \frac{gk^2}{n}(D\rho)w$$

$$- [D\{(D\mu + nd\mu')(D^2 + k^2)w\} - 2k^2 D(\mu + n\mu')Dw] = 0 \qquad ...(11)$$

## 3. TWO SUPERPOSED RIVLIN-ERICKSEN FLUIDS SEPARATED BY A HORIZONTAL BOUNDARY

Here we consider the case when two Rivlin-Ericksen superposed fluids of uniform densities $\rho_1$, $\rho_2$ uniform viscosities $\mu_1$, $\mu_2$ and uniform viscoelasticities $\mu_1'$, $\mu_2'$ are separated by a horizontal boundary at $z$=0. The subscripts 1 and 2 distinguish the lower and the upper fluids, respectively. Then, in each region of constant $\rho$ constant $\mu$ and constant $\mu'$, equation (11) becomes

$$(D^2 - k^2)(D^2 - q^2)w = 0 \qquad ...(12)$$

where

$$q^2 = k^2 + \frac{n}{\nu + \nu' n}.$$

Since $w$ must vanish both when $z \to \infty$ (in the upper fluid) and $z \to -\infty$ (in the lower fluid), the general solution of equation (12) can be written as

$$w_1 = A_1 e^{+kz} + A_2 e^{+q_1 z},\ (z < 0) \qquad \text{...(13)}$$

$$w_2 = A_3 e^{-kz} + A_4 e^{-q_2 z},\ (z > 0) \qquad \text{...(14)}$$

where $A_1, A_2, A_3, A_4$ are constants of integration,

$$q_1 = \sqrt{k^2 + \frac{n}{\nu_1 + \nu_1' n}} \text{ and } q_2 = \sqrt{k^2 + \frac{n}{\nu_2 + \nu_2' n}}, \qquad \text{...(15)}$$

In writing the solutions (13) and (14), it is assumed that $q_1$ and $q_2$ are so defined that their real parts are positive.

**4. BOUNDARY CONDITIONS** The solutions (13) and (14) must satisfy certain boundary conditions. The boundary conditions to be satisfied at the interface $z = 0$ are :

$$w, \qquad \text{...(16)}$$

$$Dw, \qquad \text{...(17)}$$

and

$$(\mu + \mu' n)(d^2 + k^2)w, \qquad \text{...(18)}$$

must be continuous.

Integrating equation (11) across the interface $z = 0$, we obtain another condition

$$[\rho_2 Dw_2 - \rho_1 Dw_1]_{z=0} - [\tfrac{1}{n}(\mu_2 + \mu_2' n)(D^2 - k^2)Dw_2 - \frac{1}{n}(\mu_1 + \mu_1' n)(D^2 - k^2)Dw_1]_{z=0}$$

$$= -\frac{gk^2}{n^2}[\rho_2 - \rho_1]w_0 - \frac{2k^2}{n}(\mu_2 + n\mu_2' - \mu_1 - n\mu_1')(Dw)_0 \qquad \text{...(19)}$$

where $w_0$, $(Dw)_0$ are the common values of $w_1$, $w_2$ and $Dw_1$, $Dw_2$ at $z = 0$.

**5. DISPERSION RELATION AND DISCUSSION** Applying the boundary conditions (16)-(19) to the solutions (13) and (14), we obtain

$$A_1 + A_2 = A_3 + A_4\ (= w_0) \qquad \text{...(20)}$$

$$kA_1 + q_1 A_2 = -kA_3 - q_2 A_4\ (= Dw_0). \qquad \text{...(21)}$$

$$(\mu_1 + \mu_1' n)\,[2k^2 A_1 + (q_1^2 + k^2)A_2] = (\mu_2 + \mu_2' n)$$

$$[2k^2 A_3 + (q_2^2 + k^2)A_4]\ \ (= \mu + \mu' n)\,(d^2 + k^2)w_0) \qquad \text{...(22)}$$

$$[-k\rho_2A_3-\rho_2q_2A_4-k\rho_1A_1-\rho_1A_1-\rho_1q_1A_2]-\frac{1}{n}(\mu_2-\mu_2'n)(-q_2)$$

$$(q_2^2-k^2)A_4-\frac{1}{n}(\mu_1+\mu_1'n)q_1(q_1^2-k^2)A_2]+\frac{gk^2}{2n^2}(\rho_2-\rho_1)$$

$$(A_1+A_2+A_3+A_4)+\frac{k^2}{n}(\mu_2+n\mu_2'-\mu_1-n\mu_1')$$

$$(kA_1+q_1A_2-kA_3-q_2A_4)=0 \qquad \text{...(23)}$$

Eliminating the constant $A_1$, $A_2$, $A_3$, $A_4$ from equations (20)-(23), we obtain

$$\begin{vmatrix} 1 & 1 & -1 & -1 \\ k & q_1 & k & q_2 \\ 2k^2(\mu_1+\mu_1'n) & \begin{array}{l}[(\mu_1+\mu_1'n)\\(q_1^2+k^2)]\end{array} & -2k^2(\mu_2+\mu_2'n) & \begin{array}{l}[-(\mu_2+\mu_2'n)\\(q_2^2+k^2)]\end{array} \\ \begin{array}{l}[-\alpha_1+\frac{R}{2}\\+\frac{k^2}{n}(\nu_2\alpha_2+\\+n\nu_2'\alpha_2-\\-\nu_1\alpha_1-n\nu_1'\alpha)]\end{array} & \begin{array}{l}[\frac{R}{2}+\frac{k}{n}(\nu_2\alpha_2+\\+n\nu_2'\alpha_2-\upsilon_1\alpha_1'-\\-n\nu_1'\alpha_1)q_1]\end{array} & \begin{array}{l}[-\alpha_2+\frac{R}{2}\\-\frac{k^2}{n}(\nu_2\alpha_2+\\+n\nu_2'\alpha_2-\\-\nu_1\alpha_1-n\nu_1'\alpha_1)]\end{array} & \begin{array}{l}[\frac{R}{2}-\frac{k}{n}(\nu_2\alpha_2+\\+n\nu_2'\alpha_2-\nu_1\alpha_1\\-n\nu_1'\alpha_1)q_2]\end{array} \end{vmatrix}=0 \qquad \text{...(24)}$$

where $\alpha_1=\dfrac{\rho_1}{\rho_1+\rho_2}$, $\alpha_2=\dfrac{\rho_2}{\rho_1+\rho_2}$, $R=\dfrac{gk}{n^2}(\alpha_2-\alpha_1)$.

The determinant can be reduced by substracting the first column from the second, the third column from the fourth and adding the first column to the third. By this procedure, we obtain

$$\begin{vmatrix} q_1-k & 2k & q_2-k \\ \begin{array}{l}[(\nu_1\alpha_1+n\nu_1'\alpha_1)\\(q_1^2-k^2)]\end{array} & \begin{array}{l}[-2k^2(\nu_2\alpha_2+n\nu_2'\alpha_2\\-\nu_1\alpha_1-n\nu_1'\alpha_1)]\end{array} & \begin{array}{l}[-(\nu_2\alpha_2+n\nu_2'\alpha_2)\\(q_2^2-k^2)]\end{array} \\ \begin{array}{l}[\frac{k}{n}(\nu_2\alpha_2+n\nu_2'\alpha_2\\-\nu_1\alpha_1-n\nu_1'\alpha_1)\\(q_1-k)+\alpha_1]\end{array} & R-1 & \begin{array}{l}[-\frac{k}{n}(\nu_2\alpha_2+n\nu_2'\alpha_2\\-\nu_1\alpha_1-n\nu_1'\alpha_1)\\(q_2-k)+\alpha_2]\end{array} \end{vmatrix}=0. \qquad \text{...(25)}$$

Evaluating the determinant (25), we obtain the following characteristic equation

$$
\begin{aligned}
&(q_1 - k)[-2k^2 (\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1) \{-\tfrac{k}{n} (\nu_2\alpha_2 \\
&+ n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1) (q_2 - k) + \alpha_2\} + (R-1) (\nu_2\alpha_2 \\
&+ n\nu_2'\alpha_2) (q_2^2 - k^2)] - 2k\, [(\nu_1\alpha_1 + n\nu_1'\alpha_1)(q_1^2 - k^2) \\
&\{-\tfrac{k}{n} (\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1) (q_2 - k) + \alpha_2\} + \\
&+ (\nu_2\alpha_2 + n\nu_2'\alpha_2) (q_2^2 - k^2) (\tfrac{k}{n} (\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1) \\
&(q_1 - k) + \alpha_1\}] + (q_2 - k)\, [(\nu_1\alpha_1 + n\nu_1'\alpha_1)(q_1^2 - k^2)(R-1) \\
&+ 2k^2 (\nu_1\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1) \{ \tfrac{k}{n} (\nu_2\alpha_2 + n\nu_2'\alpha_2 \\
&- \nu_1\alpha_1 - n\nu_1'\alpha_1) (q_1 - k) + \alpha_1\}] = 0. \qquad \text{...(26)}
\end{aligned}
$$

The dispersion relation (26) is quite complicated as the values of $q_1$ and $q_2$ involve square roots. We therefore make the assumption that the two fluids are of high viscosity and high viscoelasticity. Under this assumption, we have

$$q = k\left[1 + \frac{n}{k^2(\nu + \nu' n)}\right]^{1/2} = k + \frac{n}{2k(\nu + \nu' n)}, \qquad \text{...(27)}$$

so that

$$q_1 - k = \frac{n}{2k(\nu_1 + \nu_1' n)} \text{ and } q_2 - k = \frac{n}{2k(\nu_2 + \nu_2' n)} \qquad \text{...(28)}$$

Substituting the values of $q_1 - k$ and $q_2 - k$ from (28) in equation (26) and putting $\nu_1 = \nu_2 = \nu$, $\nu_1' = \nu_2' = \nu'$ (the case of equal kinematic viscosities and equal kinematic viscoelasticities, for mathematical simplicity), we obtain the following dispersion relation

$$(1 + 2k^2\nu')n^2 + (2k^2\nu)n - gk(\alpha_2 - \alpha_1) = 0. \qquad \text{...(29)}$$

For the potentially stable arrangement $\alpha_1 > \alpha_2$ equation (29) does not involve any change of sign and so does not allow any positive root. The system is therefore stable. This result is also true when both the fluids are viscous (Chandrasekhar [2]) or Oldroydian viscoelastic (Sharma [7]).

For the unstable configuration $\alpha_2 > \alpha_1$, there is at least one change of sign in equation (29) and so equation (29) has one positive root. The occurence of positive root implies that the system is unstable. The potentially unstable arrangement, therefore, remains unstable for the Rivlin-Ericksen elastico-viscous fluid.

We now examine the behaviour of growth rates with respect to kinematic viscosity and kinematic viscoelasticity analytically. Since $\alpha_2 > \alpha_1$ equation (29) has one positive root, let $n_0$ denote the positive root. Then

$$(1+2k^2\nu')n_0^2 + (2k^2\nu)n_0 - gk(\alpha_2-\alpha_1) = 0. \qquad \text{...(30)}$$

To study the behaviour of growth rates with respect to kinematic viscosity and kinematic viscoelasticity, we examine the natures of $\frac{dn_0}{d\nu}$ and $\frac{dn_0}{d\nu'}$ analytically.

If follows from equation (30) that

$$\frac{dn_0}{d\nu} = -\frac{k^2 n_0}{(1+2k^2\nu')n_0 + \nu k^2}, \qquad \text{...(31)}$$

and

$$\frac{dn_0}{d\nu'} = -\frac{k^2 n_0^2}{(1+2k^2\nu')n_0 + \nu k^2}. \qquad \text{...(32)}$$

From (31) and (32), we find that the growth rate decreases with the increase in kinematic viscosity as well as with the increase in kinematic viscoelasticity.

## REFERENCES

[1] P.K. Bhatia, Rayleigh-Taylor instability of two viscous superposed conducting fluid, *Nuovo Cimento B(II)* **19** (1974), 161- 169.

[2] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Clarendon Press, Oxford, 1961.

[3] R.A. Gerwin, Hydromagnetic surface waves in a conducting liquid surrounded by a flowing gas, *Phys. Fluids* **11** (1968), 1699-1708.

[4] J.G. Oldroyd, Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids, *Proc. Roy. Soc. London Ser. A* **245** (1958) 278-279.

[5] P.H. Roberts, The effect of a vertical magnetic field on Rayleigh-Taylor instability, *Astrophys. J.* **137** (1963), 679-689.

[6] R.C. Sharma and K.C. Sharma, Rayleigh-Taylor instability of two viscoelastic superposed fluids, *Acta Phys. Hungar,* **45** (1978), 213-220.

[7] R.C. Sharma, Instability of the plane interface between two viscoelastic superposed conducting fluids, *J.Math. Phys. Sci.* **12** (1978), 603-613.

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# A GENERAL FRACTIONAL INTEGRAL OPERATOR, A GENERAL CLASS OF MULTIVARIABLE POLYNOMIALS AND THE MULTIVARIABLE *H*-FUNCTION

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## ABSTRACT

The main object of this paper is to derive a fractional integral operator (involving a generalized polynomial set) of the product of a general class of multivariable polynomials and the *H*-function of several complex variables. On account of general nature of the operator, the multivariable polynomials and the *H*-function of several complex variables, our formula can be shown to yield a number of interesting (known or new) results. Some new special cases of the formula are mentioned briefly.

## 1. INTRODUCTION

### (a) General Fractional Integral Operator

The familier fractional integral operator (FIO) is defined and represented as follows :

$$ {}_cR_x^{\mu}\{f(x)\} = \frac{1}{\Gamma(\mu)}\int_c^x (x-t)^{\mu-1} f(t)\,dt, \; Re(\mu) > 0 \qquad ...(1.1)$$

The *FIO* (1.1) defines the classical Riemann-Liouville *FIO* of order $\mu$ when $c = 0$ and when $c \to \infty$ it may be identified with the definition of the familier Weyl *FIO* of order $\mu$. A number of generalizations of *FIO* are proposed recently. Agrawal, Pareek and Saigo [1,p. 55, Eq. (1.2)] have recently introduced and studied a generalization of *FIO* (1.1) which is defined by representing in the following manner :

$$ {}_cR_{x;\, r,\, s,\, q;\, m,\, k,\, l}^{\rho;\, \alpha,\, \beta,\, \tau;\, A,\, B,\, \mu}\{f(x)\}$$

$$= \frac{1}{\Gamma(\mu)} \int_c^x (x-t)^{\mu-1} S_n^{\alpha,\beta,\tau}[z(1-\tfrac{t}{x})^{\rho}; r, s, q, A, B, m, k, l]\, f(t)\, dt \qquad ...(1.2)$$

where $\mathrm{Re}(\mu) > 0$ and $S_n^{\alpha,\beta,\tau}[z]$ stands for the generalized polynomial set due to Raizada [6, p.64, Eq. (2.1.8)].

The explicit form of this generalized polynomial set ([6,p.71, Eq. (2.3.4)] : see also [1, p.56, Eq. (1.5)]) is

$$S_n^{\alpha,\beta,\tau}[x; r, s, q, A, B, m, k, l] = B^{qn} x^{l(m+n)} (l-\tau x^r)^{sn}$$

$$l^{m+n} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \sum_{j=0}^{m+n} \sum_{i=0}^{j} \frac{(-1)^j (-j)_i (\alpha)_j (-v)_{u)} (-\alpha-qn)_i}{i!\, j!\, u!\, v!\, (1-\alpha-j)_i}$$

$$\left(-\frac{\beta}{\tau}-sn\right)_v \left(\frac{i+k+ru}{l}\right)_{m+n} \left(\frac{-\tau x^r}{1-\tau x^r}\right)^v \left(\frac{Ax}{B}\right)^j \qquad ...(1.3)$$

In this paper we shall study the following special case of *FIO* (1.2) defined and represented in the following manner :

$$ {}_cR_{x:r,q;m,k,l}^{\rho;\alpha,\beta,0;1,0,\mu}\{f(x)\} = \frac{1}{\Gamma(\mu)} \int_x^c (x-t)^{\mu-1}$$

$$S_n^{\alpha,\beta,0}[z(1-t/x)^{\rho}; r, q, 1, 0, m, k, l]\, f(t)\, dt \qquad ...(1.4)$$

where $Re(\mu) > 0$ and

$$S_n^{\alpha,\beta,0}[x; r, q, 1, 0, m, k, l] = x^{qn+l(m+n)}\, l^{m+n}$$

$$\sum_{v=0}^{m+n} \sum_{u=0}^{v} \frac{(-v)_u}{v!\, u!} \left(\frac{\alpha+qn+k+ru}{l}\right)_{m+n} (\beta x^r)^v \qquad ...(1.5)$$

It may be pointed out here that *FIO* (1.4) is sufficiently general in nature as it involves a generalized polnomial set which unifies and extends a large number of classical polynomials introduced and studied by various research workers such as Chatterjea [2], Gould and Hopper [3], Krall and Frink [4] and Singh and Shrivastava [8], etc.

### (b) A General class of multivariable Polynomials

Srivastava and Garg [10,p.685] introduced and studied a general class of multivariable polynomial in the following manner

$$S_n^{m_1,\dots,m_s}[x_1,\dots,x_s]$$

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$$= \sum_{k_1, \ldots, k_s = 0}^{m_1 k_1 + \ldots + m_s k_s \le n} (-n)_{m_1 k_1 + \ldots + m_s k_s} A(n; k_1, \ldots, k_s) \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_s^{k_s}}{k_s!} \qquad \ldots(1.6)$$

where $m_1, \ldots, m_s$ are arbitrary positive integers and the coefficients $A(n; k_1, \ldots, k_s)$ $(n, k_i \ge 0, i = 1, \ldots, s)$ are arbitrary constant real or complex.

### (c) The Multivariable *H*-function

The multivariable $H$-function occurring in this paper has been defined by Srivastava and Panda [13]. We shall use the following contracted notation (Srivastava et al. [12, p.251, Eq.(C.1)] :

$$H[z_1, \ldots, z_p]$$

$$= H^{O, N; M_1, N_1; \ldots; M_p, N_p}_{P, Q; P_1, Q_1; \ldots; P_p, Q_p} \left[ \begin{matrix} z_1 \\ \vdots \\ z_p \end{matrix} \middle| \begin{matrix} (a_j; A_j', \ldots, A_j^{(p)})_{1, P}; \\ (b_j; B_j', \ldots, B_j^{(p)})_{1, Q}; \end{matrix} \right.$$

$$\left. \begin{matrix} (c_j', C_j')_{1, P_1}; \ldots; (c_j^{(p)}, C_j^{(p)})_{1, P_p} \\ (d_j', D_j')_{1, Q_1}; \ldots; (d_j^{(p)}, D_j^{(p)})_{1, Q_p} \end{matrix} \right] \qquad \ldots(1.7)$$

to denote the $H$-function of $p$-complex variables here all the capital letters are assumed to be positive real numbers for the standardization purposes; the definition of multivariable $H$-function given by (1.7) will, however, be meaningful even if some of these quantities are zero. The details of this function can be found in paper and book referred to above.

## 2. MAIN RESULT

$${}_cR^{\rho; \alpha, \beta, 0; 1, 0, \mu}_{x; r, q; m, k, l} \{[(x-a)^{v} + \xi]^{\sigma} (x-a)^{\lambda}$$

$$S_{n'}^{m_1, \ldots, m_s} [y_1 (x-a)^{\lambda_1} \{(x-a)^{v} + \xi\}^{-\tau_1}, \ldots, y_s (x-a)^{\lambda_s} \{(x-a)^{v} + \xi\}^{-\tau_s}]$$

$$H[z_1 (x-a)^{\rho_1} \{(x-a)^{v} + \xi\}^{-\sigma_1}, \ldots, z_p (x-a)^{\rho_p} \{(x-a)^{v} + \xi\}^{-\sigma_p}]\}$$

$$= \frac{1}{\Gamma(\mu)} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \sum_{k_1, \ldots, k_s = 0}^{m_1 k_1 + \ldots + m_s k_s \le n'} \sum_{w=0}^{\infty} \sum_{\delta=0}^{\infty} \frac{(-1)^{w+\delta}}{w!\, \delta!}$$

$$\frac{B(m, n, u, v)\, D(n', k_i, y_i) \xi}{(\mu \cdot \quad \cdot \delta)}$$

$$(x-a)^{\lambda + vu \quad o + \sum_{i=1}^{s} \lambda_i k_i} (x-c)^{\mu + A + \delta} x^{-A}$$

$$H^{0, N+2 : M_1, N_1; \ldots; M_p, N_p}_{P+2, Q+2 : P_1, Q_1; \ldots; P_p, Q_p} \left[ \begin{matrix} z_1 (x-a)^{\rho_1} \xi^{-\sigma_1} \\ \vdots \\ z_p (x-a)^{\rho_p} \xi^{-\sigma_p} \end{matrix} \right|$$

$$\left(-\lambda - vw - \sum_{i=1}^{s} \lambda_i k_i; \rho_1, \ldots, \rho_p\right), \left(b_j; B_j', \ldots, B_j^{(p)}\right)_{1, Q'}$$

$$\left(1 - w + \sigma - \sum_{i=1}^{s} \tau_i k_i; \sigma_1, \ldots, \sigma_p\right), \left(a_j; A_j', \ldots, A_j^{(p)}\right)_{1, p} :$$

$$\left(\delta - \lambda - vw - \sum_{i=1}^{s} \lambda_i k_i; \rho_1, \ldots, \rho_p\right), \left(1 + \sigma - \sum_{i=1}^{s} \tau_i k_i; \sigma_1, \ldots, \sigma_p\right) :$$

$$\left. \begin{matrix} (c_j', C_j')_{1, P_1}; \ldots; (c_j^{(p)}, C_j^{(p)})_{1, P_p} \\ (d_j', D_j')_{1, Q_1}; \ldots; (d_j^{(p)}, D_j^{(p)})_{1, Q_p} \end{matrix} \right] \qquad \ldots(2.1)$$

where for convienience

$$B(m, n, u, v) = \frac{(-v)_u}{v!\, u!} \left( \frac{\alpha + qn + k + ru}{l} \right)_{m+n} l^{m+n} \beta^{v} z^{A/\rho} \ldots(2.2)$$

$$D(n', k_i, y_i) = (-n')_{m_1 k_1 + \ldots + m_s k_s} A(n'; k_1, \ldots, k_s) \frac{y_1^{k_1}}{k_1!} \cdots \frac{y_s^{k_s}}{k_s!} \ldots(2.3)$$

$$A = qn\rho + l\rho(m+n) + rv\rho \qquad \ldots(2.4)$$

and (in addition to the appropriate convergence and existence conditions [12,p.252]), we assume that

$$\min\{v, \rho, \lambda, \sigma, \lambda_i, \tau_i, \rho_j, \sigma_j\} > 0 \; (i = 1, \ldots, s, j = 1, \ldots, p),$$

$$|\arg (x-a)^{v}/\xi| < \pi, c \geq a \geq 0,$$

$$\mathrm{Re}\left[ \lambda + \sum_{i=1}^{s} \lambda_i k_i + \sum_{j=1}^{p} \rho_j \zeta_i \right] > -1 \; (\zeta_j = \min_{1 \leq k \leq m_j} [Re(d_k^{(j)}/D_k^{(j)})],$$

Re $[\mu + qn\rho + l\rho(m+n) + rv\rho] > 0$ $(v = 1, \ldots, m+n)$ and the multiple series occuring on right hand side of (2.1) is absolutely convergent.

**Derivation of formula (2.1):**

Using the defintion of *FIO* (1.4) in (2.1), we find that

$$\text{L.H.S. of (2.1)} = \frac{1}{\Gamma(\mu)} \int_c^x (x-t)^{\mu-1} \{(t-a)^{\lambda} [(t-a)^{v} + \xi]^{\sigma}$$

$$S_n^{\alpha,\,\beta,\,0}\,[z\,(1-t/x);\, r, q, 1, 0, m, k, l]$$

$$S_{n'}^{m_1,\,\dots,\,m_s}\,[y_1(t-a)^{\lambda_1}\,\{(t-a)^{v}+\xi\}^{-\tau_1},\dots,\, y_s\,(t-a)^{\lambda s}\,\{(t-a)^{v}+\xi\}^{-\tau_s}]$$

$$H\,[z_1(t-a)^{\rho_1}\,\{(t-a)^{v}+\xi\}^{-\sigma_1},\dots,\, z_p(t-a)^{\rho_p}\,(t-a)^{v}+\xi\}^{-\sigma_p}]\,\}dt \qquad \text{...(2.5)}$$

Now replacing in (2.5) the generalized polynomial set $S_n^{\alpha,\,\beta,\,0}\,[z\,(1-t/x)^{\rho} : r, q, 1, 0, m, k, l]$ the general class of multivariable polynomials,

$$S_{n'}^{m_1,\dots,m_s}\,[y_1(t-a)^{\lambda_1}\,\{(t-a)^{v}+\xi\}^{-\tau_1},\dots y_s\,(t-a)^{\lambda_s}\,\{(t-a)^{v}+\xi\}^{-\tau_s}]$$

by their series given by (1.5) and (1.6) respectively and multivariable $H$-function by its Mellin-Barnes contour integral given by (1.7) and interchanging the order of integeration and summation, we get

L.H.S. of (2.1)

$$= \frac{1}{\Gamma(\mu)} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \sum_{k_1,\dots,k_s=0}^{m_1k_1+\dots+m_sk_s \le n'} x^{-A}$$

$$B(m, n, u, v)\, D(n', k_i, y_i)\, \frac{1}{(2\pi i)^p} \int_{L_1} \dots, \int_{L_p} \phi\,(\eta_1, \dots, \eta_p)$$

$$\prod_{j=1}^{p} \{\theta_j\,(\eta_j)(z_j)^{\eta_j}\}'\, \{\int_c^x (x-t)^{\mu+A-1}\,(t-a)^{\lambda+\sum_{i=1}^{s}\lambda_i k_i+\sum_{j=1}^{p}\rho_j\eta_j}$$

$$[(t-a)^{v}+\xi]^{\sigma-\sum_{i=1}^{s}\tau_i k_i-\sum_{j=1}^{p}\sigma_j\eta_j}\, dt\}\, d\eta_1,\dots,d\eta_p \qquad \text{...(2.6)}$$

Applying the binomial expansion

$$(x+\xi)^{\lambda} = \xi^{\lambda} \sum_{w=0}^{\infty} \frac{(-1)^{w}\,(-\lambda)_w}{w!} \left(\frac{x}{\xi}\right)^{w}, \left(\left|\frac{x}{\xi}\right| < 1\right) \qquad \text{...(2.7)}$$

Changing the order of integration and summation and using the well-known formula (cf,[7,p.22]) :

$$\int_c^x (x-t)^{\mu-1}\,(t-a)^{\lambda}\,dt$$

$$= (x-c)^{\mu}\,(x-a)^{\lambda} \sum_{\delta=0}^{\infty} \frac{(-1)^{\delta}\,\Gamma(\lambda+1)}{(\mu+\lambda)\,\delta!\,\Gamma(\lambda-\delta+1)} \left(\frac{x-c}{x-a}\right)^{\delta} \qquad \text{...(2.8)}$$

where $\mathrm{Re}\,(\mu) > 0$, $\mathrm{Re}(\lambda) > -1$ and $c \ge a \ge 0$, we obtain

$$\text{L.H.S. of } (2.1) = \frac{1}{\Gamma(\mu)} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \sum_{k_1,\ldots,k_s=0}^{m_1k_1+\ldots+m_sk_s \le n'} \sum_{w=0}^{\infty} \sum_{\delta=0}^{\infty}$$

$$B(m,n,u,v)\, D(n',k_i,y_i)\, \frac{(-1)^{w+\delta}}{w!\,\delta!}\, \xi^{\sigma - w - \sum_{i=1}^{s} \tau_i k_i}$$

$$x^{-A}\,(\mu + A + \delta)^{-1}\,(x-c)^{\mu+\delta+A}\,(x-a)^{\lambda + vw - \delta + \sum_{i=1}^{s} \lambda_i k_i}$$

$$\left(\frac{1}{2\pi i}\right)^{p} \int_{L_1} \ldots \int_{L_p} \frac{\Gamma\left(w - \sigma + \sum_{i=1}^{s} \tau_i k_i + \sum_{j=1}^{p} \sigma_j \eta_j\right)}{\Gamma\left(-\sigma + \sum_{i=1}^{s} \tau_i k_i + \sum_{j=1}^{p} \sigma_j \eta_j\right)}$$

$$\frac{\Gamma\left(1 + \lambda + vw + \sum_{i=1}^{s} \lambda_i k_i + \sum_{j=1}^{p} \rho_j \eta_j\right)}{\Gamma\left(1 - \delta + \lambda + vw + \sum_{i=1}^{s} \lambda_i k_i + \sum_{j=1}^{p} \rho_j \eta_j\right)}\, \phi(\eta_1,\ldots,\eta_p)$$

$$\left[\prod_{j=1}^{p} \theta_j(\eta_j)\, z_j^{\eta_j}\right] \xi^{-\sum_{i=1}^{p} \sigma_j \eta_j}\,(x-a)^{\sum_{j=1}^{p} \rho_j \eta_j}\; ; d\eta_1,\ldots,d\eta_p \qquad \ldots(2.9)$$

Interpreting the multiple Mellin-Barnes contour integral as an $H$-function of p-variables, we led finally to the fractional integral formula (2.1) under the (sufficient) conditions of validity stated already with (2.1).

## 3. SPECIAL CASES

The importance of our generalized fractional integral formula lies in its manifold generality. First of all our *FIO* (1.4) involves the generalized polynomial set which can be reduced to a large number of polynomials (see e.g.[6]). Secondly, the general class of multi-variable polynomials involved in (2.1) can be reduced to a large spectrum of polynomials of one and several variables. Thirdly, the formula (2.1) can be suitably applied to derive the corresponding results involving a remarkably wide variety of potentially useful functions (or product of several such functions) which are expressible in terms of the *E*, *G* and *H* functions of one, two or more variables. Our main result thus provides unification and extension of various (known or new) results on *FIO*. For the sake of illustration, we mention the following interesting special cases of (2.1):

(*i*) Taking $a=c$ in (2.1) and using the well-known Gauss summation theorem to sum the $\delta$-series, we get after a little simplification

$$
{}_{c}R_{x;\, r,\, q;\, m,\, k,\, l}^{\rho;\, \alpha,\, \beta,\, 0;\, 1,\, 0,\, \mu}\ \{(x-c)^{\lambda}\, [(x-c)^{v}+\xi]^{\sigma}
$$

$$
S_{n'}^{m_1,\ldots,m_s}\, [y_1(x-c)^{\lambda_1}\, \{(x-c)^{v}+\xi\}^{-\tau_1},\ldots,y_s(x-c)^{\lambda_s}\, \{(x-c)^{v}+\xi\}^{-\tau_s}]
$$

$$
H[z_1(x-c)^{\rho_1}\, \{(x-c)^{v}+\xi\}^{-\sigma_1},\ \ldots,\ z_p\, (x-c)^{\rho_p}\, \{(x-c)^{v}+\xi\}^{-\sigma_p}]\}
$$

$$
=\frac{1}{\Gamma(\mu)}\sum_{v=0}^{m+n}\ \sum_{u=0}^{v}\ \sum_{k_1,\ldots,k_s=0}^{m_1k_1+\ldots+m_sk_s\le n'}\ \sum_{w=0}^{\infty}\frac{(-1)^{w}}{w!}B(m,n,u,v)
$$

$$
D(n',k_i,y_i)\,\Sigma\,(\mu+A)x^{-A}\,\xi^{\sigma-w-\sum_{i=1}^{s}\tau_ik_i}\,(x-c)^{\mu+\lambda+A+vw+\sum_{i=1}^{s}\lambda_ik_i}
$$

$$
H_{P+2,\, Q+2\, :\, P_1,\, Q_1\, ;\ldots;\, P_p,\, Q_p}^{0,\, N+2\, :\, M_1,\, N_1\, ;\ldots;\, M_p,\, N_p}\left[\begin{array}{c} z_1\, (x-c)^{\rho_1}\, \xi^{-\sigma_1} \\ \vdots \\ z_p\, (x-c)^{\rho_p}\, \xi^{-\sigma_p} \end{array}\right|
$$

$$
(-\lambda-vw-\sum_{i=1}^{s}\lambda_ik_i\, ;\, \rho_1,\ldots,\rho_p),
$$

$$
(b_j\, ;\, B_j',\ldots,B_j^{(p)})_{1,\, Q}
$$

$$
(1-w+\sigma-\sum_{i=1}^{s}\tau_ik_i\, ;\, \sigma_1,\ldots,\sigma_p),\ (a_j\, ;\, A_j',\ldots,A_j^{(p)})_{1,\, p}\, :
$$

$$
(-\mu-A-\lambda-vw-\sum_{i=1}^{s}\lambda_ik_i\, ;\, \rho_1,\ldots,\rho_p),\ (1+\sigma-\sum_{i=1}^{s}\tau_ik_i\, ;\, \sigma_1,\ldots,\sigma_p)
$$

$$
\left.\begin{array}{c} (c_j',\, C_j')_{1,\, P_1}\, ;\ldots;\, (c_j^{(p)},\, C_j^{(p)})_{1,\, P_p} \\ (d_j',\, D_j')_{1,\, Q_1}\, ;\ldots;\, (d_j^{(p)},\, D_j^{(p)})_{1,\, Q_p} \end{array}\right] \qquad \ldots(3.1)
$$

**(*ii*) Further taking $a=0$ in (2.1), we get**

$$
{}_{c}R_{x;\, r,\, q;\, m,\, k,\, l}^{\rho;\, \alpha,\, \beta,\, 0;\, 1,\, 0,\, \mu}
$$

$$
\{x^{\lambda}\, (x^{v}+\xi)^{\sigma}\, S_{n'}^{m_1,\ldots,m_s}\, [y_1x^{\lambda_1}\, (x^{v}+\xi)^{-\tau_1},\ldots,y_s\, x^{\lambda_s}\, (x^{v}+\xi)^{-\tau_s}]
$$

$$
H[z_1\, x^{\rho_1}\, (x^{v}+\xi)^{-\sigma_1},\ldots,z_p\, x^{\rho_p}\, (x^{v}+\xi)^{-\sigma_p}]\}
$$

$$
=\frac{1}{\Gamma(\mu)}\sum_{v=0}^{m+n}\ \sum_{u=0}^{v}\ \sum_{k_1,\ldots,k_s=0}^{m_1k_1+\ldots+m_sk_s\le n'}\ \sum_{w=0}^{\infty}\ \sum_{\delta=0}^{\infty}\frac{(-1)^{w+\delta}}{w!\ \delta!\ (\mu+A+\delta)}
$$

$$B(m, n, u, v)\, D(n', k_i, y_i)\, \xi^{\sigma - \sum_{i=1}^{s} \tau_i k_i - w}\, (x-c)^{\mu + A + \delta})\, x^{\lambda + vw - \delta - A + \sum_{i=1}^{s} \tau_i k_i}$$

$$H^{0, N+2 : M_1, N_1 ;...; M_p, N_p}_{P+2, Q+2 : P_1, Q_1 ;...; P_p, Q_p} \left[ \begin{array}{c|} z_1 x^{\rho_1} \xi^{-\sigma_1} \\ \vdots \\ z_p x^{\rho_p} \xi^{-\sigma_p} \end{array} \right.$$

$$(-\lambda - vw - \sum_{i=1}^{s} \lambda_i k_i ; \rho_1, ..., \rho_p),$$

$$(b_j ; B_j', ..., B_j^{(p)})_{1, Q},$$

$$(1 - w + \sigma - \sum_{i=1}^{s} \tau_i k_i ; \sigma_1, ..., \sigma_p), (a_j ; A_j', ..., A_j^{(p)})_{1, p} :$$

$$(\delta - \lambda - vw - \sum_{i=1}^{s} \lambda_i k_i ; \rho_1, ..., \rho_p), (1 + \sigma - \sum_{i=1}^{s} \tau_i k_i ; \sigma_1, ..., \sigma_p);$$

$$\left. \begin{array}{l} (c_j', C_j')_{1, P_1} ;...; (c_j^{(p)}, C_j^{(p)})_{1, P_p} \\ (d_j', D_j')_{1, Q_1} ;...; (d_j^{(p)}, D_j^{(p)})_{1, Q_p} \end{array} \right] \qquad ...(3.2)$$

(*iii*) Again taking $c = 0$ in (3.1), we find that

$$R^{\rho; \alpha, \beta, 0; 1, 0, \mu}_{x; r, q; m, k, l} \{x^{\lambda} (x^{v} + \xi)^{\sigma}$$

$$S^{m_1, ..., m_s}_{n'} [y_1 x^{\lambda_1} (x^{v} + \xi)^{-\tau_1}, ..., y_s x^{\lambda s} (x^{v} + \xi)^{-\tau_s}]$$

$$H[z_1 x^{\rho_1} (x^{v} + \xi)^{-\sigma_1}, ..., z_p x^{\rho_p} (x^{v} + \xi)^{-\sigma_p}]\}$$

$$= \frac{1}{\Gamma(\mu)} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \sum_{k_1, ..., k_s = 0}^{m_1 k_1 + ... + m_s k_s \leq n'} \sum_{w=0}^{\infty} \frac{(-1)^{w}}{w!}$$

$$\xi^{\sigma - \omega - \sum_{i=1}^{\sigma} \tau_i \kappa_i}\, x^{\mu + \lambda + v + w + \sum_{i=1}^{s} \lambda_i k_i}\, \Gamma(\mu + A)\, B(m, n, u, v)\, D(n', k_i, y_i)$$

$$H^{0, N+2 : M_1, N_1 ;...; M_p, N_p}_{P+2, Q+2 : P_1, Q_1 ;...; P_p, Q_p} \left[ \begin{array}{c|} z_1 x^{\rho_1} \xi^{-\sigma_1} \\ \vdots \\ z_p x^{\rho_p} \xi^{-\sigma_p} \end{array} \right.$$

$$(-\lambda - vw - \sum_{i=1}^{s} \lambda_i k_i ; \rho_1, ..., \rho_p)$$

$$(b_j' ; B_j', ..., B_j^{(p)})_{1, Q},$$

$$(1 - w + \sigma - \sum_{i=1}^{s} \tau_i k_i ; \sigma_1 ,..., \sigma_p), (a_j ; A_j' ,..., A_j^{(p)})_{1, P} :$$

$$(-\mu - A - \lambda - vw - \sum_{i=1}^{s} \lambda_i k_i ; \rho_1 ,..., \rho_p), (1 + \sigma - \sum_{i=1}^{s} \tau_i k_i ;\ \sigma_1,...,\sigma_p):$$

$$\left.\begin{matrix} (c_j', C_j')_{1, P_1} ;...; (c_j^{(p)}, C_j^{(p)})_{1, P_p} \\ (d_j', D_j')_{1, Q_1} ;...; (d_j^{(p)}, D_j^{(p)})_{1, Q_p} \end{matrix}\right] \qquad ...(3.3)$$

(*iv*) If we take $n' = 0$ and $\lambda_i, T_i \to 0$ $(i = l, ..., s)$ in (3.3), we get

$$R^{\rho;\, \alpha,\, \beta,\, 0;\, 1,\, 0,\, \mu}_{x;\, r,\, q;\, m,\, k,\, l} \{ x^{\lambda} (x^{v} + \xi)^{\sigma}$$

$$H[z_1 x^{\rho_1} (x^{v} + \xi)^{-\sigma_1} ,..., z_p x^{\rho_p} (x^{v} + \xi)^{-\sigma_p}]\}$$

$$= \frac{1}{\Gamma(\mu)} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \sum_{w=0}^{\infty} \frac{(-1)^{w}}{w!} B(m, n, u, v)\, \xi^{\sigma - \omega} x^{\mu + \lambda + vw} \Gamma(\mu + A)$$

$$H^{0,\, N+2\, :\, M_1,\, N_1 ;...; M_p,\, N_p}_{P+2,\, Q+2,\, :\, P_1,\, Q_1 ;...; P_p,\, Q_p} \left[ \begin{matrix} z_1 x^{\rho_1} \xi^{-\sigma_1} \\ : \\ z^{p} x^{\rho_p} \xi^{-\sigma_p} \end{matrix} \right|$$

$$(-\lambda - vw; \rho_1,..., \rho_p), (1 - w + \sigma; \sigma_1,...,\sigma_p), (a_j; A_j',...,A_j^{(p)})_{1, P} :$$

$$(b_j, B_j',..., B_j^{(p)})_{1, Q}, (-\mu - A - \lambda - vw; \rho_1,...,\rho_p), (1 + \sigma; \sigma_1, ..., \sigma_p) :$$

$$\left.\begin{matrix} (c_j', C_j')_{1, P_1} ;...; (c_j^{(p)}, C_j^{(p)})_{1, P_p} \\ (d_j', D_j')_{1, Q_1} ;...; (d_j^{(p)}, D_j^{(p)})_{1, Q_p} \end{matrix}\right] \qquad ...(3.4)$$

Further, if we put $m = n = \alpha = k = 0$ in (3.4), we arrive at a recent result due to Srivastava et al. [9,p.563, Eq.(2.1)], which contains another known result due to Srivastava and Goyal [11,p.644, Eq.(2.1)] as a special case.

The conditions of validity of the fractional integral formula (3.1) through (3.4) are obtainable fairly easily from those of their parent formula (2.1).

(*v*) Now, appealing to the known relationship (c.f. [13, p.272, Eq.(4.7)], see also [12,p.253, Eq.(C.9)]), it is not difficult to derive the following fractional integral formula as a special case of (3.4):

$$R^{\rho;\, \alpha,\, \beta,\, 0;\, 1,\, 0,\, \mu}_{x;\, r,\, q;\, m,\, k,\, l} \{x^{\lambda} (x^{v} + \xi)^{\sigma}$$

$$F^{P:P_1,\ldots,P_p}_{Q:Q_1,\ldots,Q_p}\left[\begin{array}{l}(a_j\,;A'_j,\ldots,A^{(p)}_j)_{1,P}:\\(a_j\,;B'_j,\ldots,B^{(p)}_j)_{1,Q}:\end{array}\right.$$

$$\left.\begin{array}{l}(c'_j,C'_j)_{1,P_1};\ldots;(c^{(p)}_j,C^{(p)}_j)_{1,P_p};\\(d'_j,D'_j)_{1,Q_1};\ldots;(d^{(p)}_j,D^{(p)}_j)_{1,Q_p};\end{array}z_1x^{\rho_1}(x^{\upsilon}+\xi)^{-\sigma_1},\ldots,z_px^{\rho_p}(x^{\upsilon}+\xi)^{-\sigma_p}\right]\Big\}$$

$$=\xi^{\sigma}x^{\mu+\lambda}\frac{\Gamma(\lambda+1)}{\Gamma(\mu)}\sum_{\upsilon=0}^{m+n}\sum_{u=0}^{\upsilon}B(m,n,u,\upsilon)\frac{\Gamma(\mu+A)}{\Gamma(1+\lambda+\mu+A)}$$

$$F^{P+2:P_1;\ldots;P_p;0}_{Q+2:Q_1;\ldots;Q_p;0}\left[\begin{array}{l}(1+\lambda\,;\rho_1,\ldots,\rho_p,\upsilon),(-\sigma\,;\sigma_1,\ldots,\sigma_p,1),\\(1+\lambda+\mu+A;\rho_1,\ldots,\rho_p,\upsilon),(-\sigma\,;\sigma_1,\ldots,\sigma_p,0),\end{array}\right.$$

$$(a_j\,;A'_j,\ldots,A^{(p)}_j)_{1,P}:(c'_j,C'_j)_{1,P_1};\ldots;(c^{(p)}_j,C^{(p)}_j)_{1,P_p}:$$

$$(b_j\,;B'_j,\ldots,B^{(p)}_j)_{1,Q}:(d'_j,D'_j)_{1,Q_1};\ldots;(d^{(p)}_j,D^{(p)}_j)_{1,Q_p}:$$

$$\left.z_1x^{\rho_1}\xi^{-\sigma_1},\ldots,z_px^{\rho_p}\xi^{-\sigma_p},-\frac{x^{\upsilon}}{\xi}\right]\qquad\ldots(3.5)$$

(*vi*) Further, putting $\sigma=0,\ \upsilon=\rho_1=\ldots=\rho_p=1$ and $\sigma_i\to 0$ $(i=1,\ldots,p)$ in (3.5), we get

$$R^{\rho;\,\alpha,\,\beta,\,0;\,1,\,0,\,\mu}_{x;\,r,\,q;\,m,\,k,\,l}\left[x^{\lambda}F^{P:P_1;\ldots;P_p}_{Q:Q_1;\ldots;Q_p}\left|\begin{array}{l}(a_j\,;A'_j,\ldots,A^{(p)}_j)_{1,P}:\\(b_j;B'_j,\ldots,B^{(p)}_j)_{1,Q}:\end{array}\right.\right.$$

$$\left.\begin{array}{l}(c'_j,C'_j)_{1,P_1};\ldots;(c^{(p)}_j,C^{(p)}_j)_{1,P_p};\\(d'_j,D'_j)_{1,Q_1};\ldots;(d^{(p)}_j,D^{(p)}_j)_{1,Q_p};\end{array}z_1x,\ldots,z_px\right]$$

$$=\frac{x^{\mu+\lambda}\,\Gamma(\lambda+1)}{\Gamma(\mu)}\sum_{\upsilon=0}^{m+n}\sum_{u=0}^{\upsilon}B(m,n,u,\upsilon)\frac{\Gamma(\mu+A)}{\Gamma(1+\lambda+\mu+A)}$$

$$F^{P+1:P_1;\ldots;P_p}_{Q+1:Q_1;\ldots;Q_p}\left[\begin{array}{l}(1+\lambda\,;1,\ldots,1):\\(1+\lambda+\mu+A;1,\ldots,1):\end{array}\right.$$

$$\left.\begin{array}{l}(c'_j,C'_j)_{1,P_1};\ldots;(c^{(p)}_j,C^{(p)}_j)_{1,P_p};\\(d'_j,D'_j)_{1,Q_1};\ldots;(d^{(p)}_j,D^{(p)}_j)_{1,Q_p};\end{array}z_1x,\ldots,z_px\right]\qquad\ldots(3.6)$$

(*vii*) From (3.6), we obtain easily the following formulas involving Lauricella hypergeometric functions of $p$ variables :

$$R^{\rho;\,\alpha,\,\beta,\,0;\,1,\,0,\,\mu}_{x;\,r,\,q;\,m,\,k,\,l}\{x^{\lambda}\,[a_1,\ldots,a_p,b_1,\ldots,b_p\,;1+\lambda;z_1x,\ldots,z_px]\}$$

$$= \frac{x^{\mu+\lambda}\,\Gamma(\lambda+1)}{\Gamma(\mu)} \sum_{v=0}^{m+n} \sum_{u=0}^{v} B(m,n,u,v)\, \frac{\Gamma(\mu+A)}{\Gamma(1+\lambda+\mu+A)}$$

$$F_B^{(p)}\,[a_1,\ldots,a_p, b_1,\ldots,b_p\,;\,1+\lambda+\mu+A; z_1x,\ldots,z_px]\} \qquad \ldots(3.7)$$

$$R^{\rho;\,\alpha,\,\beta,\,0;\,1,\,0,\,\mu}_{x;\,r,\,q,\,m,\,k,\,l}\,\{x^{\lambda} \prod_{i=1}^{p} (1-z_ix)^{-\mu_i}\}$$

$$= \frac{x^{\mu+\lambda}\,\Gamma(\lambda+1)}{\Gamma(\mu)} \sum_{v=0}^{m+n} \sum_{u=0}^{v} B(m,n,u,v)\, \frac{\Gamma(\mu+A)}{\Gamma(1+\lambda+\mu+A)}$$

$$F_D^{(p)}\,[1+\lambda, b_1,\ldots,b_p\,;\,1+\lambda+\mu+A; z_1x,\ldots,z_px] \qquad \ldots(3.8)$$

If in (3.8), we put $\rho = m = n = v = \alpha = k = 0$ and replace $\lambda$ by $\lambda - 1$, $\mu$ by $\mu - \lambda$, we arrive at a result due to Srivastava and Goyal [11,p.649, Eq. (3.6)] which for $p = 2$ reduces to another known result due to Lavoie et al. [5,p.260].

## REFERENCES

[1] R. Agrawal, R.S. Pareek and M. Saigo, A general fractional integral formula, *J. Frac. Cal.* **7** (1995), 55-60.

[2] S.K. Chatterjea, Quelques foncutions gènèratrices des polynômes d'Hermite du point de vue de l'algbere de Lie, *C.R. Acad. Sci. Paris Ser. A-B* **268** (1969), A 600-A 602.

[3] H.W. Gould and A.T. Hopper, Operational formulas connected with two generalizations of Hermite polynomials, *Duke Math. J.* **29** (1962), 51-63.

[4] H.L. Krall and O. Frink, A new class of orthogonal polynomials : The Bessel polynomials, *Trans. Amer. Math. Soc.* **65** (1949), 100- 115.

[5] J.L. Lavoie, T.J. Osler and R. Tremblay, Fractional derivatives and special functions, *SIAM Rev.* **18** (1976), 240-268.

[6] S.K. Raizada, *A Study of Unified Representation of Special Functions of Mathematical Physics and Their Use in Statistical and Boundary Value Problems,* Ph.D. thesis, Bundelkhand Univ., India, 1991.

[7] B. Ross, A brief history and exposition of the fundamental theory of fractional calculus, *Fractional Calculus and its Applications* (B. Ross, Ed.), pp. 1-36, Springer-Verlag, Berlin/Heidelberg/New York, 1975.

[8] R.P. Singh and K.N. Srivastava, A note on generalization of Laguerre and Humbert polynomials, *Ricerca (Napoli)* (2), **14** (1963), 11-21; *errata, ibid.* **15** (1964), 63.

[9] H.M. Srivastava, R.C.S. Chandel and P.K. Vishwakarma, Fractional derivatives of certain generalized hypergeometric functions of several variables, *J.Math.Anal.Appl.* **184**(1994), 560-572.

[10] H.M. Srivastava and M. Garg, Some integrals involving a general class of polynomials and the multivariable H-function, *Rev. Roumaine Phys.* **32** (1987), 685-692.

[11] H.M. Srivastava and S.P. Goyal, Fractional derivatives of the H- function of several variables, *J.Math.Anal.Appl.* **112**(1985), 641- 651.

**[12] H.M. Srivastava, K.C. Gupta and S.P. Goyal, *The H-functions of one and two variables with Applications,* South Asian Publishers, New Delhi/Madras, 1982.**

**[13] H.M. Srivastava and R. Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials, *J. Reine Angew. Math.* 283/284 (1976), 265-274.**

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# A SOLUTION OF THE PARTIAL DIFFERENTIAL EQUATION OF HEAT CONDUCTION IN A ROD UNDER THE ROBIN CONDITION

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## ABSTRACT

The role of special functions and their applications is becoming increasingly of much significance in technical applications. Space research and nuclear reactor also give rise to several problems of application of special functions.

This paper has been designed to discuss the application of certain products involving the multivariable *H*-function and two general class of polynomials in establishing a solution of the partial differential equation

$$\frac{\partial \theta}{\partial t} = \mu \frac{\partial^2 \theta}{\partial x^2}$$

related to a problem of heat conduction in a rod under the Robin condition. The result gives a number of known particular cases on specializing the parameters and may prove to be useful in several interesting situations appearing in the literature on mathemaitcal analysis, applied mathematics and mathematical physics.

**1. INTRODUCTION.** As an example of the application of the multivariable *H*-function in applied mathematics, the problem of obtaining a solution of a problem of flow of heat in a uniform rod under third kind of boundary conditions, i.e., the problem of heat conduction in a uniform rod under the Robin condition (at zero temperature with radiation at the ends inside the medium) will be consider in this problem. If the thermal coefficient is constant and their is no source of thermal energy, then a function $u(x, t)$ representing the temperature in a rod $(0 \le x \le L)$ must satisfy the heat equation

$$\frac{\partial\theta}{\partial t} = \mu \frac{\partial^2\theta}{\partial x^2} \quad \text{...(1.1)}$$

and the initial conditions

$$\theta(x, 0) = f(x) \quad \text{...(1.2)}$$

$$\frac{\partial\theta}{\partial x}(0, t) - h\theta(0, t) = 0 \quad \text{...(1.3)}$$

$$\frac{\partial\theta}{\partial x}(L, t) + h\theta(L, t) = 0;\ h > 0 \quad \text{...(1.4)}$$

Here we shall consider

$$f(x) = \left(\sin\frac{\pi x}{L}\right)^{\sigma-1} S_n^m\left[y\left(\sin\frac{\pi x}{L}\right)^{2\rho}\right] S_{n'}^{m'}\left[y'\left(\sin\frac{\pi x}{L}\right)^{2\rho'}\right]$$

$$H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\begin{pmatrix} z_1(\sin\frac{\pi x}{L})^{2k_1} \\ \vdots \\ z_r(\sin\frac{\pi x}{L})^{2k_r} \end{pmatrix} \quad \text{...(1.5)}$$

(For the multivariable $H$-function and a general class of polynomials, see [7] and [6], respectively).

The expression

$$e^{-\mu E_q^2 t}(m \cos E_q x + n \sin E_q x) \quad \text{...(1.6)}$$

satisfies the equation (1.1), Equation (1.6) also satisfies (1.3) and (1.4) on the condition that

$$E_q n - hm = 0 \quad \text{...(1.7)}$$

and $E_q(n \cos E_q L - m \sin E_q L) + h(m \cos E_q L + n \sin E_q L) = 0$ (1.8)

From (1.6) and (1.7), we get

$$A/B = Eq/h \quad \text{...(1.9)}$$

and $\tan E_q L = 2E_q h/(E_q^2 - h^2)$ ...(1.10)

where $E_q$ is the $q$th positive root of (1.14). Then the solution of our problem takes the following elegant from

$$\theta(x, t) = \sum_{q=1}^{\infty} R_q \left(\cos E_q x + \frac{h}{E_q}\sin E_q x\right) e^{-\mu E_q^2 t} \quad \text{...(1.11)}$$

The following results will be required :

$$\int_0^L (\sin \frac{\pi x}{L})^{\sigma - 1} \sin \frac{E_p \pi x}{L} dx$$

$$= L.2^{1-\sigma} \sin \frac{E_p \pi}{2} \frac{\Gamma(\sigma)}{\Gamma(\frac{\sigma \pm E_p + 1}{2})}, R(\sigma) > 0 \quad ...(1.12)$$

$$\int_0^l (\sin \frac{\pi x}{L})^{\sigma - 1} \cos \frac{E_p \pi x}{L} dx$$

$$= L.2^{1-\sigma} \cos \frac{E_p \pi}{2} \frac{\Gamma(\sigma)}{\Gamma(\frac{\sigma \pm E_p + 1}{2})}, R(\sigma) > 0 \quad ...(1.13)$$

$$\int_0^L (\cos E_q x + \frac{h}{E_q} \sin E_q x)(\cos E_p x + h_l E_p \sin E_p x)\, dx$$

$$= \begin{cases} 2(E_q^2)^{-1} [(E_q^2 + h^2)L + 2h], & p = q \\ 0, & p \neq q \end{cases} \quad ...(1.14)$$

where $E_q$ is positive root of the non-algebraic equation

$$\tan EL = \frac{2hE}{(E^2 - h^2)} \quad ...(1.15)$$

We shall require the following definition (cf. Srivastava [6], p.1, eqn.(1.1)]

$$S_n^m [x] = \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s} x^s, n = 0, 1, 2, \ldots \quad ...(1.16)$$

where $m$ is arbitrary positive integer and the coefficients $F_{n,s}$ $(n, s \geq 0)$ are arbitrary constants, real or complex. By suitably specializing the coefficients $F_{n,s}$ the polynomials $S_n^m [x]$ can be reduced to the well-known classical orthogonal polynomials such as Jacobi, Hermite, Legendre, Tchebycheff and Laguerre polynomials, etc.

## 2. THE MAIN INTEGRAL FORMULAE

$$\int_0^L (\sin \frac{\pi x}{L})^{\sigma - 1} \sin \frac{E_p \pi x}{L} S_n^m \left[ y (\sin \frac{\pi x}{L})^{2\rho} \right]$$

$$S_{n'}^{m'}\left[ y' (\sin \frac{x\pi}{L})^{2\rho'} \right] H^{0,\, \lambda\, :\, (u',\, v')\, ;\dots;\, (u^{(r)},\, v^{(r)})}_{A,\, C\, :\, [B',\, D']\, ;\dots;\, [B^{(r)},\, D^{(r)}]} \left( \begin{matrix} z_1 (\sin \frac{\pi x}{L})^{2k_1} \\ \vdots \\ z_r (\sin \frac{\pi x}{L})^{2k_r} \end{matrix} \right) dx$$

$$= L.2^{1 - \sigma - 2\rho s - 2\rho' s'} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,\, s}\, y^s$$

$$\sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!} F_{n',\, s'}\, y'^{s'} \sin \frac{E_p \pi}{2}$$

$$H^{0,\, \lambda + 1\, :\, (u',\, v')\, ;\dots;\, (u^{(r)},\, v^{(r)})}_{A+1,\, C+2\, :\, [B',\, D']\, ;\dots;\, [B^{(r)},\, D^{(r)}]}$$

$$\left( \begin{matrix} [1 - \sigma - 2\rho s - 2\rho' s' : 2k_1, \dots, 2k_r],\ [(a) : \theta', \dots, \theta^{(r)}] : \\ [(c) : \psi', \dots, \psi^{(r)}],\ [\, \frac{1 - \sigma - 2\rho s - 2\rho' s' \pm E_p}{2} : k_1, \dots, k_r] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : \phi'] \,;\dots;\, [(b^{(r)}) : \phi^{(r)}] \,;\, z_1\, 4^{-k_1} \\ \vdots \\ [(d') : \delta'] \,;\dots;\, [(d^{(r)}) : \delta^{(r)}] \,;\, z_r\, 4^{-k_r} \end{matrix} \right) \quad \dots(2.1)$$

where $\rho > 0$, $\rho' > 0$, $k_i > 0$, $\mathrm{Re}\,(\sigma + \sum_{i=1}^{r} k_i d_j^{(i)}/\delta_j^{(i)}) > 0$, $1 \le i \le r$, $1 \le j \le u^{(i)}$, $|\arg (z_i)| < T_i\, \pi/2$, $T_i > 0$, $m, m'$ are arbitrary positive integers and the coefficients $F_{n,\, s}\, (n, s \ge 0)$ and $F_{n',\, s'}\, (n', s' \ge 0)$ are arbitrary constants, real or complex.

$$\int_0^L (\sin \frac{\pi x}{L})^{\sigma - 1} \cos \frac{E_p\, \pi x}{L}\, S_n^m \left[ y\, (\sin \frac{\pi x}{L})^{2\rho} \right]$$

$$S_{n'}^{m'} \left[ y' (\sin \frac{\pi x}{L})^{2\rho'} \right] H^{0,\, \lambda\, :\, (u',\, v')\, ;\dots;\, (u^{(r)},\, v^{(r)})}_{A,\, C\, :\, [B',\, D']\, ;\dots;\, [B^{(r)},\, D^{(r)}]} \left( \begin{matrix} z_1 (\sin \frac{\pi x}{l})^{2k_1} \\ \vdots \\ z_r (\sin \frac{\pi x}{l})^{2k_r} \end{matrix} \right) dx$$

$$= L.2^{1 - \sigma - 2\rho s - 2\rho' s'} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,\, s}\, y^s$$

$$\sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!} F_{n',\, s'}\, y'^{s'} \cos \frac{E_p \pi}{2}$$

$$H^{0,\, \lambda+1\, :\, (u',\, v')\, ;\ldots;\, (u^{(r)},\, v^{(r)})}_{A+1,\, C+2\, :\, [B',\, D']\, ;\ldots;\, [B^{(r)},\, D^{(r)}]} \left( \begin{array}{l} [1-\sigma-2\rho s-2\rho' s' : 2k_1, \ldots, 2k_r],\, [(a) : \theta', \ldots, \theta^{(r)}] : \\ [(c) : \psi', \ldots, \psi^{(r)}],\, \left[ \dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_p}{2} : k_1, \ldots, k_r \right] : \\ \qquad [(b') : \phi'] ;\ldots; [(b^{(r)}) : \phi^{(r)}] ;\ z_1\, 4^{-k_1} \\ \qquad \qquad \qquad \vdots \\ \qquad [(d') : \delta'] ;\ldots; [(d^{(r)}) : \delta^{(r)}] ;\ z_r\, 4^{-k_r} \end{array} \right) \qquad \ldots(2.2)$$

where $\rho > 0$, $\rho' > 0$, $k_i > 0$, $\mathrm{Re}\,(\sigma + \sum_{i=1}^{r} k_i\, d_j^{(r)}/d_j^{(i)}) > 0$, $1 \leq i \leq r$, $1 \leq j \leq u^{(i)}$, $|\arg(z_i)| < T_i\, \pi/2$, $T_i > 0$, $m, m'$ are arbitrary positive integers and the coefficients $F_{n,s}$ $(n, s \geq 0)$ and $F_{n',s'}$ $(n', s' \geq 0)$ are arbitrary constants, real or complex. The integral formulae in (2.1) and (2.2) can be derived by making use of the definition of the multivariable $H$-function (Srivastava and Panda [7] and Srivastava [8], p.251) and of the results in (1.12), (1.13) and (1.16) respectively.

**3. SOLUTION OF THE PROBLEM** The solution of the problem to be established

$$\theta\,(x, t) = \frac{2L}{2^{\sigma+2\rho s+2\rho' s'-1}} \sum_{q=1}^{\infty} \sum_{s=0}^{[n/m]} \sum_{s'=0}^{[n'/m']} \frac{(-n)_{ms}}{s!}$$

$$F_{n,s}\, y^{s}\, \frac{(-n')_{m's'}}{s'!}\, F_{n',s'}\, y'^{s'}\, \frac{E_q^2}{[(E_q^2+h^2)\, L + 2h]}\, e^{-\mu E_q^2 t}$$

$$(\cos E_q x + \frac{h}{E_q} \sin E_q x)\, (\cos \frac{E_q \pi}{2} + \frac{h}{E_q} \sin \frac{E_q \pi}{2})$$

$$H^{0,\, \lambda+1\, :\, (u',\, v')\, ;\ldots;\, (u^{(r)},\, v^{(r)})}_{A+1,\, C+2\, :\, [B',\, D']\, ;\ldots;\, [B^{(r)},\, D^{(r)}]} \left( \begin{array}{l} [1-\sigma-2\rho s-2\rho' s' : 2k_1, \ldots, 2k_r],\, [(a) : \theta', \ldots, \theta^{(r)}] : \\ [(c) : \psi', \ldots, \psi^{(r)}],\, [\, \dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_q}{2} : k_1, \ldots, k_r] : \\ \qquad [(b') : \phi'] ;\ldots; [(b^{(r)}) : \phi^{(r)}] ;\ z_1\, 4^{-k_1} \\ \qquad \qquad \qquad \vdots \\ \qquad [(d') : \delta'] ;\ldots; [(d^{(r)}) : \delta^{(r)}] ;\ z_r\, 4^{-k_r} \end{array} \right) \qquad \ldots(3.1)$$

where $\rho > 0$, $\rho' > 0$, $k_i > 0$, $\text{Re}\,(\sigma + \sum_{i=1}^{r} k_i d_j^{(i)}/\delta_j^{(i)} > 0$, $1 \le i \le r$, $1 \le j \le u^{(i)}$, $|arg\,(z_i)| < T_i\,\pi/2$, $T_i > 0$, $m, m'$ are arbitrary positive integers and the coefficients $F_{n,s}\,(n, s \ge 0)$ and $F_{n',s'}\,(n', s' > 0)$ are arbitrary constants, real or complex. Derivation of (3.1).

If $t = 0$ then from (1.5) and (1.11), we have

$$(\sin \frac{\pi x}{L})^{\sigma - 1}\, S_n^m\left[\, y\,(\sin \frac{\pi x}{L})^{2\rho} \right] S_{n'}^{m'}\left[\, y'\,(\sin \frac{\pi x}{L})^{2\rho'} \right]$$

$$H^{0,\,\lambda\,:\,(u',\,v')\,;\dots;\,(u^{(r)},\,v^{(r)})}_{A,\,C\,:\,[B',\,D']\,;\dots;\,[B^{(r)},\,D^{(r)}]}\left(\begin{matrix} z_1\,(\sin \frac{\pi x}{L})^{2k_1} \\ \vdots \\ z_r\,(\sin \frac{\pi x}{L})^{2k_r} \end{matrix}\right)$$

$$= \sum_{q=1}^{\infty} R_q\,(\cos E_q x + \frac{h}{E_q} \sin E_q x) \qquad \dots(3.2)$$

Multiply both sides of (3.2) by

$$(\cos E_p x + \frac{h}{E_p} \sin E_p x)$$

then integrate with respect to $x$ from 0 to $L$, and use (1.4), (2.1) and (2.2) to obtain

$$R_p = 2LE_p^2\,[(E_p^2 + h^2)L + 2h]^{-1}\, 2^{1 - \sigma - 2\rho s - 2\rho' s'}$$

$$\sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!}\, F_{n,s}\, y^s \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!}\, F_{n',s'}\, y'^{s'}$$

$$(\cos \frac{E_p \pi}{2} + \frac{h}{E_p} \sin \frac{E_p \pi}{2})\, H^{0,\,\lambda + 1\,:\,(u',\,v')\,;\dots;\,(u^{(r)},\,v^{(r)})}_{A + 1,\,C + 2\,:\,[B',\,D']\,;\dots;\,[B^{(r)},\,D^{(r)}]}$$

$$\left(\begin{matrix} [1 - \sigma - 2\rho s - 2\rho' s' : 2k_1, \dots, 2k_r],\ [(a) : \theta', \dots, \theta^{(r)}] : \\ [(c) : \psi', \dots, \psi^{(r)},\ [\frac{1 - \sigma - 2\rho s - 2\rho' s' \pm E_p}{2} : k_1, \dots, k_r] : \end{matrix}\right.$$

$$\left.\begin{matrix} [(b') : \phi'] \,;\dots;\, [(b^{(r)}) : \phi^{(r)}] \,;\, z_1\, 4^{-k_1} \\ \vdots \\ [(d') : \delta'] \,;\dots;\, [(d^{(r)}) : \delta^{(r)}] \,;\, z_r\, 4^{-k_r} \end{matrix}\right) \qquad \dots(3.3)$$

Substitute the values of $R_p$ from (3.3) in (3.2), the desired solution (3.1) is established.

## 4. SPECIAL CASES AND APPLICATIONS

(*i*) For $r = 1$, the results in (2.1), (2.2) and (3.1) reduce to the following results involving Fox's $H$-function [5]

$$\int_0^L \left(\sin\frac{\pi x}{L}\right)^{\sigma-1} \sin\frac{E_p\pi x}{L}\, S_n^m\left[y\left(\sin\frac{\pi x}{L}\right)^{2\rho}\right]$$

$$. S_{n'}^{m'}\left[y'\left(\sin\frac{\pi x}{L}\right)^{2\rho'}\right] H_{B,D}^{u,v}\left[z\left(\sin\frac{\pi x}{L}\right)^{2k} \middle| \begin{matrix}[(b):\phi]\\ [(d):\delta]\end{matrix}\right]dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s}\, y^s \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!}$$

$$.F_{n',s}\, y'^{s'} \sin\frac{E_p\pi}{2}\, H_{B+1,D+2}^{u,v+1}\left[z\,4^{-k}\middle| \begin{matrix}[1-\sigma-2\rho s-2\rho' s' : 2k],\ [(b):\phi]\\ [(d):\delta],\ [\dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_p}{2} : k]\end{matrix}\right] \quad ...(4.1)$$

where $\rho > 0$, $\rho' > 0$, $k > 0$, Re $(\sigma + kd_j/\delta_j > 0$, $1 \le j \le u$, $m$ and $m'$ are arbitrary positive integers and the coefficients $F_{n,s}$ $(n, s \ge 0)$ and $F_{n',s'}$ $(n', s' \ge 0)$ are arbitrary constants, real or complex.

$$\left(T = \sum_{1}^{v} \phi_i - \sum_{v+1}^{B} \phi_i + \sum_{1}^{u} \delta_i - \sum_{u+1}^{D} \delta_i\right)$$

$$\int_0^L \left(\sin\frac{\pi x}{L}\right)^{\sigma-1} \cos\frac{E_p\pi x}{L}\, S_n^m\left[y\sin\frac{\pi x}{L}\right)^{2\rho}\right]$$

$$. S_{n'}^{m'}\left[y'\left(\sin\frac{\pi x}{L}\right)^{2\rho'}\right] H_{B,D}^{u,v}\left[z\left(\sin\frac{\pi x}{L}\right)^{2k} \middle| \begin{matrix}[(b):\phi]\\ [(d):\delta]\end{matrix}\right]dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s}\, y^s F_{n,s} \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!}$$

$$.F_{n',s}\, y'^{s'} \cos\frac{E_p\pi}{2}\, H_{B+1,D+2}^{u,v+1}\left[z4^{-k}\middle| \begin{matrix}[1-\sigma-2\rho s-2\rho' s' : 2k],\ [(b):\phi]\\ [(d):\delta],\ [\dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_p}{2} : k]\end{matrix}\right] \quad ...(4.2)$$

where $\rho > 0$, $\rho' > 0$, $k > 0$, Re $(\sigma + k\, d_j/\delta_j > 0$, $1 < j \le u$,

$|\arg(z)| < T, \pi/2$, $T > 0$, $m$ and $m'$ are arbitrary positive integers and the coefficients $F_{n,s}$ $(n, s \ge 0)$ and $F_{n',s'}$ $(n', s' \ge 0)$ are arbitrary constants, real or complex.

$$\theta(x, t) = \frac{2L}{2^{\sigma + 2\rho s + 2\rho' s' - 1}} \sum_{q=1}^{\infty} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s} y^s$$

$$\sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!} F_{n',s'} y'^{s'} \quad \frac{E_q^2}{[(E_q^2 + h^2) L + 2h]} e^{-\mu E_q^2 t} \quad .$$

$$(\cos E_q x + \frac{h}{E_q} \sin E_q x)(\cos \frac{E_q \pi}{2} + \frac{h}{E_q} \sin \frac{E_q \pi}{2})$$

$$H^{u, v+1}_{B+1, D+2}\left[ z4^{-k} \left| \begin{matrix} [1 - \sigma - 2\rho s - 2\rho' s' : 2k], [(b) : \phi] \\ [(d) : \delta] \; [\dfrac{1 - \sigma - 2\rho s - 2\rho' s' \pm E_q}{2} : k] \end{matrix} \right. \right] \quad ...(4.3)$$

where $\rho > 0$, $\rho' > 0$, $k > 0$, $\text{Re}(\sigma + kd_j/\delta_j) > 0$, $1 \leq j \leq u$, $|\arg(z)| < T\pi/2$, $T > 0$, $m$ and $m'$ are arbitrary positive integer and the coefficients $F_{n,s}$ $(n, s \geq 0)$ and $F_{n',s'}$ $(n', s' \geq 0)$ are arbitrary constants real or complex.

(*ii*) Letting $n \to 0$, $n' \to 0$ and $y = y' = 1$ the results in (2.1), (2.2), (3.1), (4.1), (4.2) and (4.3) follow as particular cases of results recently obtained by Chaurasia ([1], eqns. (16) and (17), p.79, eqn. (18), p.80, eqn. (21), p.81 and eqns. (22) and (23), p.82).

(*iii*) Taking $n' \to 0$, $y = y' = 1$ the results in (2.1), (2.2), (3.1), (4.1), (4.2) and (4.3) follow as particular cases of the results recently obtained by Chaurasia and Girdhari Lal in [2].

(*iv*) By applying our results derived in (2.1), (2.2) and (2.3) to the case of Hermite polynomials ([10], p.106, eqn. (5.5.4) and [9], p.158) by setting

$$S_n^2 [x] \to x^{n/2} H_n (\frac{1}{2\sqrt{x}})$$

in which case $m = 2$, $F_{n,s} = (-1)^s$ and also letting $m' = 2$, $F_{n',s'} = (-1)^{s'}$, we have the following interesting consequences of the main result :

$$\int_0^L (\sin \frac{\pi x}{L})^{\sigma - 1} \sin \frac{E_p \pi x}{L} \left[ y (\sin \frac{\pi x}{L})^{2\rho} \right]^{n/2}$$

$$. H_n \left[ \frac{1}{2\sqrt{y (\sin \frac{\pi x}{L})^{2\rho}}} \right] \left[ y' (\sin \frac{\pi x}{L})^{2\rho'} \right]^{n'/2}$$

$$. H_{n'}\left[\frac{1}{2\sqrt{y'(\sin\frac{\pi x}{L})^{2\rho'}}}\right] H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \begin{pmatrix} z_1(\sin\frac{\pi x}{L})^{2k_1}\\ \vdots \\ z_r(\sin\frac{\pi x}{L})^{2k_r}\end{pmatrix} dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'} \sum_{s=0}^{[n/2]} \frac{(-n)_{2s}}{s!}(-y)^s \sum_{s'=0}^{[n'/2]}\frac{(-n')_{2s'}}{s'!}$$

$$(-y')^{s'} \sin\frac{E_p\pi}{2}\, H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];\ldots;[B^{(r)},D^{(r)}]}$$

$$\left( \begin{matrix} [1-\sigma-2\rho s-2\rho' s':2k_1,\ldots,2k_r],[(a):\theta',\ldots,\theta^{(r)}]: \\ [(c):\psi',\ldots,\psi^{(r)}],[\frac{1-\sigma-2\rho s-2\rho' s'\pm E_p}{2}:k_1,\ldots,k_r]: \end{matrix} \right.$$

$$\left. \begin{matrix} (b'):\phi'];\ldots;[(b^{(r)}):\phi^{(\rho)}];z_1 4^{-k_1}\\ \vdots \\ [(d'):\delta;\ldots;[(d^{(r)}):\delta^{(r)}];z_r 4^{-k_r}\end{matrix}\right) \qquad \ldots(4.4)$$

which holds true under the same conditions as those required for (2.1)

$$\int_0^L (\sin\frac{\pi x}{L})^{\sigma-1} \cos\frac{E_p\pi x}{L}\left[y(\sin\frac{\pi x}{L})^{2\rho}\right]^{n/2}$$

$$.H_n\left[\frac{1}{2\sqrt{y(\sin\frac{\pi x}{L})^{2\rho}}}\right]\left[y'(\sin\frac{\pi x}{L})^{2\rho'}\right]^{n'/2} H_{n'}\left[\frac{1}{2\sqrt{y'(\sin\frac{\pi x}{L})^{2\rho'}}}\right]$$

$$H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \begin{pmatrix} z_1(\sin\frac{\pi x}{L})^{2k_1}\\ \vdots \\ z_r(\sin\frac{\pi x}{L})^{2k_r}\end{pmatrix} dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'} \sum_{s=0}^{[n/2]} \frac{(-n)_{2s}}{s!}(-y)^s \sum_{s'=0}^{[n'/2]}\frac{(-n')_{2s'}}{s'!}$$

$$(-y')^{s'} \cos\frac{E_p\pi}{2}\, H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];\ldots;[B^{(r)},D^{(r)}]}$$

$$\left( \begin{array}{l} [1-\sigma-2\rho s-2\rho' s' : 2k_1, \ldots, 2k_r], [(a) : \theta', \ldots, \theta^{(r)}] : \\ [(c) : \psi', \ldots, \psi^{(r)}], [\dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_p}{2} : k_1, \ldots, k_r] : \end{array} \right.$$

$$\left. \begin{array}{ll} (b') : \phi'] ; \ldots ; [(b^{(r)}) : \phi^{(\rho)}] ; & z_1 \, 4^{-k_1} \\ & \vdots \\ [(d') : \delta'] ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ; & z_r \, 4^{-k_r} \end{array} \right) \quad \ldots(4.5)$$

valid under some conditions as needed for (2.2)

$$\theta(x,t) = \frac{2L}{2^{\sigma+2\rho s+2\rho' s'-1}} \sum_{q=1}^{\infty} \sum_{s=0}^{[n/2]} \sum_{s'=0}^{[n'/2']} \frac{(-n)_{2s}}{s!} (-y)^s$$

$$\frac{(-n')_{2s'}}{s'!} (-y')^{s'} \frac{E_q^2}{[(E_q^2+h^2)L+2h]} e^{-\mu E_q^2 t} (\cos E_q x + \frac{h}{E_q} \sin E_q x)$$

$$(\cos \frac{E_q \pi}{2} + \frac{h}{E_q} \sin \frac{E_q \pi}{2}) H^{0, \lambda+1 : (u', v') ; \ldots ; (u^{(r)}, v^{(r)})}_{A+1, C+2 : [B', D'] ; \ldots ; [B^{(r)}, D^{(r)}]}$$

$$\left( \begin{array}{l} [1-\sigma-2\rho s-2\rho' s' : 2k_1, \ldots, 2k_r], [(a) : \theta', \ldots, \theta^{(r)}] : \\ [(c) : \psi', \ldots, \psi^{(r)}, [\dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_q}{2} : k_1, \ldots, k_r] : \end{array} \right.$$

$$\left. \begin{array}{ll} [(b') : \phi'] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ; & z_1 4^{-k_1} \\ & \vdots \\ [(d') : \delta'] ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ; & z_r \, 4^{-k_r} \end{array} \right) \quad \ldots(4.6)$$

which holds true under the same conditions as those required for (3.1)

(v) Taking $\lambda = A_i$, $u^{(i)} = 1$, $v^{(i)} = B^{(i)}$, $D^{(i)} = D^{(i)} + 1$, $\forall \, i = 1, 2, \ldots, r$ in (2.1), (2.2) and (3.1), we get

$$\int_0^L (\sin \frac{\pi x}{L})^{\sigma-1} \sin \frac{E_p \pi x}{L} \; S_n^m [(y \frac{\pi x}{L})^{2\rho}]$$

$$. S_{n'}^{m'} \left[ y' (\sin \frac{\pi x}{L})^{2\rho'} \right] F^{A : B' ; \ldots ; B^{(r)}}_{C : D' ; \ldots ; D^{(r)}} \left( \begin{array}{l} [1-(a) : \theta', \ldots, \theta^{(r)}] : \\ [1-(c) : \psi', \ldots, \psi^{(r)}] : \end{array} \right.$$

$$\left. \begin{array}{ll} [1-(b') : \phi'] ; \ldots ; [1-(b^{(r)}) : \phi^{(r)}] ; & -z_1 (\sin \frac{\pi x}{L})^{2k_1} \\ [1-(d') : \delta'] ; \ldots ; [1-(d^{(r)}) : \delta^{(r)}] ; & \vdots \\ & -z_r (\sin \frac{\pi x}{l})^{2k_r} \end{array} \right) dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s}\, y^{s} \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!}$$

$$.F_{n',s'}\, y'^{s'} \sin\frac{E_p\pi}{2} \frac{\Gamma(\sigma+2\rho s+2\rho' s')}{\Gamma\left(\frac{\sigma+2\rho s+2\rho' s' \pm E_p+1}{2}\right)}$$

$$.F^{A+1:B';\dots;B^{(r)}}_{C+2:D';\dots;D^{(r)}}\left(\begin{matrix}[\sigma+2\rho s+2\rho' s' : 2k_1,\dots,2k_r],\\ [1-(c):\psi',\dots,\psi^{(r)}],\end{matrix}\right.$$

$$[1-(a):\theta',\dots,\theta^{(r)}] : [1-(b'):\phi'];\dots;[1-(b^{(r)}):\phi^{(r)}];$$

$$\left[\frac{\sigma+2\rho s+2\rho' s' \pm E_p+1}{2} : k_1,\dots,k_r\right]:[1-(d'):\delta'];\dots;[1-(d^{(r)}):\delta^{(r)}]$$

$$\left. -z_1 4^{-k_1},\dots,-z_r 4^{-k_r}\right) \qquad \dots(4.7)$$

valid under the same conditions as obtainable from (2.1).

$$\int_0^L \left(\sin\frac{nx}{L}\right)^{\sigma-1} \cos\frac{E_p\pi x}{L}\, S_n^m\left[y\left(\sin\frac{\pi x}{L}\right)^{2\rho}\right]$$

$$S_{n'}^{m'}\left[y'\left(\sin\frac{\pi x}{L}\right)^{2\rho'}\right] F^{A:b';\dots;B^{(r)}}_{C:d';\dots;D^{(r)}}\left(\begin{matrix}[1-(a):\theta',\dots,\theta^{(r)}:\\ [1-(c):\psi',\dots,\psi^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[1-(b'):\phi'];\dots;[1-(b^{(r)}):\phi^{(r)}];\\ [1-(d'):\delta'];\dots;[1-(d^{(r)}):\delta^{(r)}];\end{matrix}\ \begin{matrix}-z_1\left(\sin\frac{\pi x}{L}\right)^{2k_1}\\ \vdots\\ -z_r\left(\sin\frac{\pi x}{L}\right)^{2k_r}\end{matrix}\right)dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s}\, y^{s} \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!}$$

$$.F_{n',s'}\, y'^{s'} \cos\frac{E_p\pi}{2} \frac{\Gamma(\sigma+2\rho s+2\rho' s')}{\Gamma\left(\frac{\sigma+2\rho s+2\rho' s' \pm E_p+1}{2}\right)}$$

$$.F^{A+1:B';\dots;B^{(r)}}_{C+2:D';\dots;D^{(r)}}\left(\begin{matrix}[\sigma+2\rho s+2\rho' s': 2k_1,\dots,2k_r],\ [1-(a):\theta',\dots,\theta^{(r)}]:\\ [1-(c):\psi',\dots,\psi^{(r)}],\left[\frac{\sigma+2\rho s+2\rho' s' \pm E_p+1}{2}:k_1,\dots,k_r\right]:\end{matrix}\right.$$

$$\left.\begin{matrix}[1-(b'):\phi'];\dots;[1-(b^{(r)}):\phi^{(r)}];\\ [1-(d'):\delta'];\dots;[1-(d^{(r)}):\delta^{(r)}];\end{matrix}\ \begin{matrix}-z_1 4^{-k_1}\\ \vdots\\ -z_r 4^{-k_r}\end{matrix}\right) \qquad \dots(4.8)$$

valid under the same conditions as need for (2.2)

$$\theta(x,t)=\frac{2L}{2^{\sigma+2\rho s+2\rho' s'-1}}\sum_{q=1}^{\infty}\sum_{s=0}^{[n/m]}\sum_{s'=0}^{[n'/m']}\frac{(-n)_{ms}}{s!}$$

$$.F_{n,s}\,y^{s}\frac{(-n')_{m',s'}}{s'!}\,F_{n',s'}\,y'^{s'}\frac{E_q^2}{[(E_q^2+h^2)\,L+2h]}\,e^{-\mu E_q^2 t}$$

$$(\cos E_q x+\frac{h}{E_q}\sin E_q x)\,(\cos\frac{E_q\pi}{2}+\frac{h}{E_q}\sin\frac{E_q\pi}{2})$$

$$\frac{\Gamma(\sigma+2\rho s+2\rho' s')}{\Gamma\left(\dfrac{\sigma+2\rho s+2\rho' s'\pm E_q+1}{2}\right)}\;\frac{\prod_{j=1}^{A}\Gamma(1-a_j)\prod_{j=1}^{B'}\Gamma(1-b'_j)\ldots\prod_{j=1}^{B^{(r)}}\Gamma(1-b_j^{(r)})}{\prod_{j=1}^{C}\Gamma(1-c_j)\prod_{j=1}^{D'}\Gamma(1-d'_j)\ldots\prod_{j=1}^{D^{(r)}}\Gamma(1-d_j^{(r)})}$$

$$.\,F^{A+1:B';\ldots;B^{(r)}}_{C+2:D';\ldots;D^{(r)}}\left(\begin{matrix}[\sigma+2\rho s+2\rho' s':2k_1,\ldots,2k_r],\,[1-(a):\theta',\ldots,\theta^{(r)}]: \\ [1-(c):\psi',\ldots,\psi^{(r)}],\left[\dfrac{\sigma+2\rho s+2\rho' s'\pm E_q+1}{2}:k_1,\ldots,k_r\right]\end{matrix}\right.$$

$$\left.\begin{matrix}[1-(b'):\phi'];\ldots;[1-b^{(r)}:\phi^{(r)}]; \\ [1-(d'):\delta'];\ldots;[1-d^{(r)}:\delta^{(r)}];\end{matrix}\;\begin{matrix}-z_1\,4^{-k_1} \\ \vdots \\ -z_r\,4^{-k_r}\end{matrix}\right) \qquad \ldots(4.9)$$

valid under the same conditions as obtainable from (3.1).

(*vi*) For the Laguerre polynomials ([10]; p.101, eqn. 1.5, 1.6) and ([9], p.158) setting $S_n^1(x)\to L_n^{(\alpha')}(x)$,

in which case $m=1$ and $F_{n,s}=\binom{n+\alpha'}{n}\frac{1}{(\alpha'+1)_s}$

and also taking $m'=1$ and $F_{n',s'}=\binom{n'+\alpha''}{n'}\frac{1}{(\alpha''+1)_{s'}}$

the results in (2.1), (2.2) and (3.1) reduce to the following formulae :

$$\int_0^L(\sin\frac{\pi x}{L})^{\sigma-1}L_n^{\alpha'}\left[y\,(\sin\frac{\pi x}{L})^{2\rho}\right]L_{n'}^{\alpha''}\left[y'\,(\sin\frac{\pi x}{L})^{2\rho'}\right]$$

$$.\sin\frac{E_p\pi x}{L}\;H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}z_1\,(\sin\frac{\pi x}{L})^{2k_1} \\ \vdots \\ z_r\,(\sin\frac{\pi x}{L})^{2k_r}\end{matrix}\right)dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'} \sum_{s=0}^{n} \binom{n+\alpha'}{n-s} \frac{(-y)^s}{s!} \sum_{s'=0}^{n'} \binom{n'+\alpha''}{n'-s'}$$

$$\cdot \frac{(-y')^{s'}}{s'!} \sin\frac{E_p\pi}{2}\; H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];\ldots;[B^{(r)},D^{(r)}]}$$

$$\left(\begin{array}{l} [1-\sigma-2\rho s-2\rho' s' : 2k_1,\ldots,2k_r],\ [(a):\theta',\ldots,\theta^{(r)}] : . \\ [(c):\psi',\ldots,\psi^{(\rho)}],\left[\dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_p}{2} : k_1,\ldots,k_r\right]: \end{array}\right.$$

$$\left.\begin{array}{r} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\ z_1\,4^{-k_1} \\ \vdots \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\ z_r 4^{-k_r} \end{array}\right) \qquad \ldots(4.10)$$

valid under the same conditions as needed for (2.1)

$$\int_0^L (\sin\frac{\pi x}{L})^{\sigma-1} \cos\frac{E_p \pi x}{L}\, L_n^{\alpha'}\left[y\,(\sin\frac{\pi x}{L})^{2\rho}\right] L_{n'}^{\alpha''}\left[y'\,(\sin\frac{\pi x}{L})^{2\rho'}\right]$$

$$H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{array}{l} z_1\,(\sin\frac{\pi x}{L})^{2k_1} \\ \vdots \\ z_r(\sin\frac{\pi x}{L})^{2k_r} \end{array}\right) dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'} \sum_{s=0}^{n} \binom{n+\alpha'}{n-s} \frac{(-y')^s}{s!} \sum_{s'=0}^{n'} \binom{n'+\alpha''}{n'-s'} \frac{(-y')^{s'}}{s'!} \cos\frac{E_p\pi}{2}$$

$$H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];\ldots;(B^{(r)},D^{(r)})]}$$

$$\left(\begin{array}{l} [1-\sigma-2\rho s-2\rho' s' : 2k_1,\ldots,2k_r],\ [(a):\theta',\ldots,\theta^{(r)}]: \\ [(c):\psi',\ldots,\psi^{(r)}],\ [\dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_p}{2} : k_1,\ldots,k_r]: \end{array}\right.$$

$$\left.\begin{array}{r} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\ z_1\,4^{-k_1} \\ \vdots \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\ z_r\,4^{-k_r} \end{array}\right) \qquad \ldots(4.11)$$

valid under the same conditions as obtainable from (2.2)

$$\theta\,(x,t) = \frac{2L}{2^{\sigma+2\rho s+2\rho' s'-1}} \sum_{q=1}^{\infty} \sum_{s=0}^{n} \binom{n+\alpha'}{n-s} \frac{(-y)^s}{s!}$$

$$\sum_{s'=0}^{n'} \binom{n'+\alpha''}{n'-s'} \frac{(-y')^{s'}}{s'!} \frac{E_q^2}{[(E_q^2+h^2)L+2h]} e^{-\mu E_q^2 t}$$

$$(\cos E_q x + \frac{h}{E_q} \sin E_q x)\,(\cos \frac{E_q \pi}{2} + \frac{h}{E_q} \sin \frac{E_q \pi}{2})$$

$$H^{0,\,\lambda+1\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A+1,\,C+2\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}$$

$$\left(\begin{matrix} [1-\sigma-2\rho s-2\rho' s' : 2k_1, \ldots, 2k_r],\ [(a) : \theta', \ldots, \theta^{(r)}] : \\ [(c) : \psi', \ldots, \psi^{(r)}],\ [\dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_q}{2} : k_1, \ldots, k_r] : \end{matrix}\right.$$

$$\left.\begin{matrix} [(b') : \phi']\,;\ldots;\,[b^{(r)} : \phi^{(r)}]\,;\, z_1 4^{-k_1} \\ \vdots \\ [(d') : \delta']\,;\ldots;\,[d^{(r)} : \delta^{(r)}]\,;\, z_r 4^{-k_r} \end{matrix}\right) \qquad \ldots(4.12)$$

valid under the same conditions as obtainable from (3.1).

The importance of our resutls lies in its manifold generality. In view of the generality of the multivariable $H$-function, on specializing the various parameters and variables, from our results, several integrals and solutions involving a remarkably wide variety of useful functions (or products of several such functions), which are expressible in terms of $E$, $F$, $G$ and $H$ functions of one and several variables.

Secondly, on suitably specializing the coefficients $F_{n,\,s}$ and also the coefficients $F_{n',\,s'}$ and making a free use of the special cases of $S_n^m\,[x]$ listed by Srivastava and Singh [9], our results can be reduced to a large number of integerals and the solution of the problem involving the products of generalized Hermite polynomials, Jacobi polynomials, Brafman polynomials, Gould-Hopper polynomials and their various combinations. Thus, the result derived in this paper would at once yield a very large number of results, involving a large variety of polynomials and various special functions occuring in the problems of mathematical analysis, applied mathematics and mathematical physics.

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## REFERENCES

[1] V.B.L. Chaurasia, Applications of the multivariable H-function in heat conduction in a rod under the Robin condition, *Jñānābha* **21** (1991), 78-82.

[2] V.B.L. Chaurasia and Girdhari Lal, Use of the *H*-function of several complex variables and a general class of polynomials in heat conduction in a rod under the Robin condition, *Indian J. Pure Appl. Math.* **23** (1992), 439-496.

[3] R.V. Churchill, *Fourier Series and Boundary Value Problems,* McGraw-Hill Book Co., New York, 1941.

[4] H.S. Carslaw and J.C. Jaeger, *Condution of Heat of Solids, Second ed.,* Oxford University Press, 1959.

[5] C. Fox, The *G*- and *H*-functions as symmetrical Fourier Kernels, *Trans. Amer. Math. Soc.* **98** (1961), 395-429.

[6] H.M. Srivastava, A contour integral involving Fox's *H*-function, *Indian J. Math,* **14** (1972), 1-6.

[7] H.M. Srivastava and R. Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials, *J. Reine Angew. Math.* **283/284** (1976), 265-274.

[8] H.M. Srivastava, K.C. Gupta and S.P. Goyal, *The H-functions of one and two variables with Applications,* South Asian Publishers, New Delhi and Madras, 1982.

[9] H.M. Srivastava and N.P. Singh, The integration of certain products of the multivariable *H*-function with a general class of polynomials, *Rend. Circ. Mat. Palermo* (2) **32** (1983), 157-187.

[10] G. Szegö, *Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ.* **23,** Fourth edition, Amer. Math. Soc., Providence, Rh ode Island, 1975.

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# EXISTENCE THEOREMS FOR COMMON SOLUTIONS OF DIFFERENTIAL EQUATIONS IN BANACH SPACE

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## ABSTRACT

The aim of the present paper is to establish the stability of the common solution of two differential equations in Banach spaces via fixed point technique.

**1. INTRODUCTION.** At present there is an abundance of literature available on the stability of differential equations with the different notions and with the different approaches. See for example. Hirsch and Smale [2], Lakshmikantham and Leela [3] and Lasalle [4]. The stability of differential equations via the fixed point technique is well-known in the literature. Recently the present author proved a common fixed point theorem for a pair of mappings in Banach spaces and which has been applied to prove the common solution of two differential equations in Banach spaces [1]. In this paper the simultaneous stability of two differential equations in Banach spaces is established. Before proving the main resutls of this paper, we first prove a fixed point theorem which will be used in the sequel. We need the following preliminaries.

Let $E$ denote a Banach space. A subset $K$ of $E$ is said to be a cone in $E$ if $(i)$ $k_r + k \le K$ $(ii)$ $\lambda k \subseteq K$, whenever $\lambda \ge 0$, and $(iii)$ $K \cap -K = \{0\}$. The relation $x \le y$ if and only if $y - x \in K$, defines a partial ordering in $E$.

**Definition 1.1 :** Two mappings $A$ and $B$ on a Banach space into itself with a order relation $\le$ induced by the closed cone $K$ in $X$, are said to be **isotone increasing** if $A(x) \le B(A(x))$ and $B(x) \le A(B(x))$ for all $x \in X$. If the reverse inequalities are satisfied, then $A$ and $B$ are called weakly **isotone decreasing** on $X$. Finally, $A$ and $B$ are called **weakly isotone** on $X$ if they are either **weakly isotone** increasing or decreasing on $X$.

**Theorem 1.1 :** Let $S$ be a non-empty, closed, convex and bounded subset of a Banach space $X$ with an order relation $\le$ induced by the closed cone $K$ in $X$ and let $A, B : S \to S$ be two completely continuous and weakly isotone mappings. Then $A$ and $B$ have a common fixed point.

**Proof :** Let $S$ be arbitrary and consider the sequence $\{x_n\}$ in $S$ defined by

$$x_0 = x, x_{2n+1} = Ax_{2n}, x_{2n+2} = BX_{2n+1} \qquad ...(1.1)$$

for $n = 0, 1, 2, ....$

Suppose that $A$ and $B$ are weakly isotone increasing on $S$. Then by nature of the sequence $\{x_n\}$, we get

$$x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \qquad ...(1.2)$$

Since $A$ and $B$ are completely continuous $A(S)$ and $B(S)$ are relatively compact and hence $\overline{A(S)}$ and $\overline{B(S)}$ are compact. As $\{x_n\} \subseteq \{x_0\} \cup \overline{A(S)} \cup \overline{B(S)}$ and $\{x_n\}$ is monotone increasing, it has a unique limit point $x^*$ in $S$ such that $\lim\limits_{n \to \infty} x_n = x^*$, and every subsequence of $\{x_n\}$ converges to the same limit point. By continuity of $A$ and $B$, we obtain

$$x^* = \lim_{n \to \infty} x_{2n+1} = \lim_{n \to \infty} A(x_{2n}) = A\left(\lim_{n \to \infty} x_{2n}\right) = Ax^*$$

$$x^* = \lim_{n \to \infty} x_{2n+2} = \lim_{n \to \infty} B(x_{2n+1}) = B\left(\lim_{n \to \infty} x_{2n+1}\right) = Bx^*$$

Thus $x^*$ is a common fixed point of $A$ and $B$. Similarly, if $A$ and $B$ are weakly isotone decreasing on $S$, then it can be shown that the sequence $\{x_n\}$ in $S$ defined by (1.1) is monotone decreasing and converges to the unique limit point which is again a common fixed point of $A$ and $B$. This completes the proof.

**Corollary 1.1 :** Let $S$ be a non-empty, closed, convex and bounded subset of a Banach space $X$ with an order relation $\leq$ induced by the closed cone $K$ in $X$ and let $A, B : S \to S$ be two weakly isotone mappings. If $A$ is continuous and $B$ is a completely continuous mapping on $S$, then $A$ and $B$ have a common fixed point.

**Proof :** Let $x \in S$ be arbitrary and consider a sequence $\{x_n\}$ in $S$ defined by (1.1).

Now suppose that $A$ and $B$ are weakly isotone increasing on $S$. Then by nature of the sequence, we get

$$x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \qquad ...(1.3)$$

Since $A$ is continuous and $B$ is completely continuous, $AoB$ is completely continuous on $S$. Therefore it follows that $AB(S)$ is relatively compact and hence $\overline{AB(S)}$ and $\overline{B(S)}$ are compact in $X$. As $\{x_n\} \subseteq \{x_0, x_1\} \cup \overline{B(S)} \cup \overline{AB(S)}$, in view of (1.3) it follows that the sequence $\{x_n\}$ converges to a unique limit point. The rest of the proof is similar to Theorem 1.1. The proof is complete.

**Corollary 1.2 :** Let $S$ be a non-empty, closed, convex and bounded subset of a Banach space $X$ with an order relation $\leq$ induced by the closed cone $K$ on $X$ and let $A, B : S \to S$ be two weakly isotone mappings. Further, if $A$ is a contraction and $B$ is completely continuous, then $A$ and $B$ have a unique common fixed point.

**Proof :** Since every contraction mapping is continuous, $A$ is continuous on $S$. By corollary 1.2, the mapping $A$ and $B$ have a common fixed point. But by the Banach contraction mapping principle, it follows that A has a unique fixed point. So $A$ and $B$ have a unique common fixed point. This completes the proof.

In the following section, we prove the simultaneous stability of two differential equations, particularly initial value problems in Banach spaces, by an applications of the results proved in this section.

## 2. STABILITY OF DIFFERENTIAL EQUATIONS

Let $R$ denote the real line and $J = [t_0, t_0 + a] \subset R$, for some $t_0$, $a \in R$, $a > 0$ a closed and bounded interval. Let $E$ denote a Banach space with norm $\| . \|_E$ and an order relation $\leq$ induced by a closed cone $K$ in it. Now consider the nonlinear differential equations (in short $DEs$) with the same initial condition (for convenience),

$$\left.\begin{aligned} x' &= f(t, x) \\ x(t_0) &= x_0 \in E \end{aligned}\right\} \qquad \text{...(2.1)}$$

and

$$\left.\begin{aligned} x' &= g(t, x) \\ x(t_0) &= x_0 \in E \end{aligned}\right\} \qquad \text{...(2.2)}$$

for $t \in J$, where $f, g \in C(J \times E, E)$.

We seek the simultaneous stability of the common solution of the $DEs$ (1.1) and (1.2) in the space $X = C(J, E)$ of all continuous $E$-valued functions on $J$. Define a norm $\| . \|$ in X by

$$\| X \| = \sup_{t \in J} \| x(t) \|_E \qquad \text{...(2.3)}$$

Clearly $X$ is a Banach space with this supremum norm. We introduce an order relation $\leq$ in $X$ as follows. Let $K$ denotes a cone in $X$ defined by

$$K = \{x \in X \mid x(t) \in K, \text{ for all } t \in J\} \qquad \text{...(2.4)}$$

Let $x, y \in X$, then by $x \leq y$ we mean $y - x \in K$, or equivalently $y(t) - x(t) \in K$ for all $t \in J$, i.e. $x(t) \leq y(t)$ for all $t \in J$. Obviously $K$ is a closed cone in $X$.

**Definition 2.1 :** The differential equations (2.1) and (2.2) are said to have a stable common solution $P$ if for $\varepsilon > 0$, there exists an $n > 0$ such that $\|x_0\| \leq n$ implies $\| p(t) \|_E \leq \varepsilon$ for all $t \in J$.

We consider the following set of assumptions :

$(A_1)$ For given $\varepsilon > 0$ there exists a $\delta > 0$, $a\delta > 1$, such that

$$\| f(t,x) - f(t,y) \|_E \le \delta \| x - y \|_E$$

and
$$\| g(t,x) - g(t,y) \|_E \le \varepsilon \| x - y \|_E$$

for all $(t, x), (t, y) \in J \times E$ whenever $\| x \|_E \le \varepsilon$, $\| y \|_E \le \varepsilon$.

$(A_2)$ $f(t, \theta) = \theta = g(t, \theta)$ for all $t \in J$, where $\theta$ is the zero vector of $E$.

$(A_3)$
$$f(t, x(t)) \le g\left( t, \int_0^t f(s, x(s))\, ds + x_0 \right)$$

and
$$g(t, x(t)) \le f\left( t, \int_{t_0}^t f(s, x(s))\, ds + x_0 \right)$$

for $(t, x) \in J \times X$ and for a fixed element $x_0 \in E$ gien in (2.1) and (2.2).

$(A_4)$ $f(t, x)$ and $g(t, x)$ are continuous in $t$, uniformly for $x \in E$ with $\| x \|_E < \infty$.

$(A_5)$ The function $f(t, x)$ satisfies the Lipschitz condition in $x$, uniformly for $t \in J$, i.e. there exists a constant $L > 0$ such that

$$\| f(t,x) - f(t,y) \|_E \le L \| x - y \|_E$$

for all $(t, x), (t, y) \in J \times E$.

**Theorem 2.1 :** Assume $(A_1) - (A_4)$. Then the *DEs* (2.1) and (2.2) have a stable common solution on $J$, whenever $\| x_0 \|_E \le b\, \varepsilon$ for some $b \in (0, 1)$.

**Proof :** Let $\in > 0$ be given and take $\delta = \dfrac{1-b}{a}$ for some $b \in (0, 1)$, which is possible in view of $a\delta < 1$. Then by $(A_1)$ and $(A_2)$, we have

$$\left. \begin{aligned} &\| f(t,x) \|_E \le \frac{1-b}{a} \varepsilon \\ \text{and} \quad &\| g(t,x) \|_E \le \frac{1-b}{a} \varepsilon \end{aligned} \right\} \qquad \text{...(2.5)}$$

for all $(t, x) \in J \times E$.

Define a subset $\overline{S}(\varepsilon)$ of the Banach space $X$ by

$$\overline{S}(\varepsilon) = \{x \in X / \| x \| \le \varepsilon\} \qquad \text{...(2.6)}$$

Clearly a subset $\overline{S}$ $(\varepsilon)$ is a closed, convex and bounded subset of the Banach space $X$. Now the *DEs* (2.1) and (2.2) are equivalent to the integral equations,

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s))\, ds\,, \quad t \in J \qquad \text{...(2.7)}$$

and
$$x(t) = x_0 + \int_{t_0}^{t} g(s, x(s))\, ds\ , \quad t \in J \qquad \text{...(2.8)}$$

respectively.

Define two operators $A$ and $B$ on $\overline{S}(\varepsilon)$ by right hand sides of the equations (2.7) and (2.8), respectively. Then the problem of a common solution of the *DEs* (2.1) and (2.2) is reduced to finding the common fixed points of the operators $A$ and $B$. We show that the operators $A$ and $B$ map $\overline{S}(\varepsilon)$ into itself. Let $x \in \overline{S}(\varepsilon)$ be arbitrary then by $(A_1) - (A_2)$, we obtain

$$\| \; Ax(t) \; \|_E \; \leq \| \; x_0 \; \|_E \; + \int_{t_0}^{t} \| \; f(s, x(s)) \; \|_E \; ds$$

$$\leq b\varepsilon + \frac{1-b}{a}\, \varepsilon . a = \varepsilon$$

for all $t \in J$ and hence $A$ maps $\overline{S}(\varepsilon)$ into itelf. Similarly $B$ also maps $\overline{S}(\varepsilon)$ into itself. Also the operators $A$ and $B$ are completely continuous on $\overline{S}(\varepsilon)$ in view of the hypotheses $(A_1) - (A_2)$ and $(A_4)$. Next we show that the operators $A$ and $B$ are weakly isotone increasing on $\overline{S}(\varepsilon)$, then by $(A_3)$

$$Ax(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \leq x_0 + \int_{t_0}^{t} g\left( s, \int_{t_0}^{s} f(\tau, x(\tau))\, d\tau + x_0 \right) ds$$

$$= x_0 + \int_{t_0}^{t} g(s, Ax(s))\, ds$$

$$= B(Ax(t))$$

for all $t \in J$, and so $Ax \leq B(Ax)$. Similarly $Bx \leq A(Bx)$ for all $x \in \overline{S}(\varepsilon)$. This shows that $A$ and $B$ are weakly isotone increasing on $\overline{S}(\varepsilon)$. Now an application of Theorem 1.1 yields that the operators $A$ and $B$ have a common fixed point in $\overline{S}(\varepsilon)$. Consequently the *DEs* (2.1) and (2.2) have a common solution in $\overline{S}(\varepsilon)$. Thus there exists a common solution $p$ of the *DEs* (2.1) and (2.2) such that $\| \; p(t) \; \|_E \; \leq \varepsilon$ for all $t \in J$, whenever $\| \; x_0 \; \|_E \; \leq b\varepsilon$ for some $b \in (0, 1)$. This, in view of defintion 2.1, implies that the *DEs* (2.1) and (2.2) have a stable common solution on $J$. This completes the proof.

**Theorem 2.2 :** Assume $(A_1) - (A_5)$. Further if $La < 1$ then the *DEs* (2.1) and (2.2) have a unique stable solution on $J$ whenver $\| \; x_0 \; \|_E \; \leq b\varepsilon$ for some $b \in (0, 1)$.

**Proof :** By Theorem 2.1, the *DEs* (2.1) and (2.2) have a stable solution on $J$. To prove the uniqueness, let $u$ and $v$ be two distinct stable common solutions of the *DEs* (2.1) and (2.2) on $J$. Then $u$ and $v$ satisfy the *DEs* (2.1) and (2.2) on $J$. Therefore, we have

$$u(t) = x_0 + \int_{t_0}^{t} f(s, u(s))\, ds,\ t \in J$$

and
$$v(t) = x_0 + \int_{t_0}^{t} f(s, v(s))\, ds\ ,\ t \in J$$

Then

$$\| u(t) - v(t) \|_E \le \int_{t_0}^{t} \| f(s, u(s)) - f(s, v(s)) \|_E\ ds$$
$$\le La\ \| u(t) - v(t) \|_E$$

or
$$\| u - v \| \le La\ \| u - v \|,$$

which is not possible since $La < 1$. Hence $u(t) = v(t)$ for all $t \in J$. This completes the proof.

**Remark 2.1 :** The assumption $(A_3)$ in Theorems 2.1 and 2.2 can be replaced by the following assumptions $(A_3)'$.

$(A_3)'$ $(i)$ $f(t, x)$ and $g(t, x)$ are non-decreasing in $x \in E$ uniformly for $t \in J$,

$(ii)$ $f(t, x) \le g(t, f(t, x))$ and $g(t, x) \le f(t, g(t, x))$ for all $(t, x) \in J \times E$

$(iii)$ $$f(t, x(t)) \le x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

and
$$g(t, x(t) \le x_0 + \int_{t_0}^{t} g(s, x(s))\, ds$$

for all $(t, x) \in J \times X$ and for a fixed element $x_0 \in E$ given in (2.1) and (2.2).

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## REFERENCES

[1] B.C. Dhage, Condensing mappings and existence theorems for common solutions of differential equations in Banach spaces, *J. Math. Anal. Appl.* (to appear).

[2] M.W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems and Linear Algebra*, Academic Press, New York, 1974.

[3] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities*, Vol. I, Academic Press, New York, 1969.

[4] J.P. Lasalle, *The Stability of Dynamical Systems*, SIAM, Philadelphia, 1976.

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# STABILITY OF STRATIFIED RIVLIN-ERICKSEN FLUID IN POROUS MEDIUM IN PRESENCE OF SUSPENDED PARTICLES AND VARIABLE MAGNETIC FIELD

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## ABSTRACT

The stability of stratified Rivlin-Ericksen viscoelastic fluid in stratified porous medium is considered, separately, in presence of suspended particles (fine dust) and variable horizontal magnetic field. A variable horizontal magnetic field stabilizes certain wave-number range whereas the system was unstable for all wave numbers in the absence of magnetic field, for unstable stratifications. The growth rates are found to decrease with the increase in kinematic viscosity, kinematic viscoelasticity and magnetic field whereas the growth rates increase with the increase in medium permeability. The suspended particles (fine dust) do not affect the stability or instability, as was the situation in their absence.

**1. INTRODUCTION** The stability of superposed fluids, under varying assumptions of hydrodynamics and hydromagnetics, has been treated in detail by Chandrasekhar [1]. The medium has been assumed to be non-porous and the fluid to be Newtonian. The flow through a porous medium has been of considerable interest in recent years particularly among geophysical fluid dynamicists. A macroscopic equation which describes incompressible flow of a Newtonian fluid of viscosity $\mu$ through a macrosocopically homogeneous and isotropic porous medium of permeability $k_1$ is the usual Darcy's equation wherein the usual viscous term in the equations of fluid motion is replaced by the resistance term $-\left(\frac{\mu}{k_1}\right)\vec{q}$, where $\vec{q}$ is the filter velocity of the fluid.

The effect of suspended particles on the stability of superposed fluids importance in geophysics and chemical engineering. Further motivation for the stability of fluids in presence of suspended particles is the fact that knowledge concerning fluid-particle mixtures is not commensurate with thier industrial and scientific importance. Scanlon and Segel [2] have considered the effect of suspended particles on the onset of Bénard convection and found that the critical Rayleigh number was reduced solely because the heat capacity of the pure gas was supplemented by that of the particles. The suspended particles were thus found to destabilize the layer. Palaniswamy and Purushotham [3] have considered the stability of shear flow of stratified fluids with fine dust and have found that the fine dust (suspended particles) increases the region of instability.

Generally the magnetic field has a stabilizing effect on the instability, but there are a few exceptions also. For example, Kent [4] has studied the effect of a horizontal magnetic field which varies in the vertical direction on the stability of parallel flows and has shown that the system is unstable under certain conditions, while in the absence of magnetic field the system is known to be stable.

Sharma [5] has studied the stability of a layer of electrically conducting Oldroyed fluid (i.e. fluid described by the Oldroyd [6] constitutive relation) heated from below in the presence of a uniform magnetic field and has found that the magnetic field has a stabilizing effect. There are many elastico-viscous fluids that cannot be characterized by Maxwell's constitutive relation or Oldroyd's constitutive relation. Rivlin-Ericksen fluid is one such class of viscoelastic fluids. Garg, Srivastava and Singh [7] have studied the drag on a sphere oscillating in conducting dusty Rivlin-Ericksen elastico-viscous liquid. It is this class of elastico-viscous fluid we are interested in.

Owing to the importance of suspended particles and variable magnetic field in chemical engineering and geophysics, the present paper considers the stability of stratified Rivlin- Ericksen fluid in stratified porous medium in the presence of suspended particles and variable horizontal magnetic field separately.

## 2. EFFECT OF VARIABLE MAGNETIC FIELD

### 2.1 PERTURBATION EQUATIONS

The initial stationary state whose stability we wish to examine is that of an incompressible, infinitely conducting, Rivlin-Ericksen fluid of variable density, kinematic viscosity and kinematic viscoelasticity arranged in horizontal strata in a porous medium of variable porosity and permeability and acted on by gravity force $\vec{g}(0, 0, -g)$ and variable horizontal magnetic field $\vec{H}(H_0(z), 0, 0)$. Consider an infinite horizontal layer of thickness $d$ bounded by the planes $z = 0$ and $z = d$.

Let $\delta\rho$, $\delta p$, $\vec{v}(u, v, w)$ and $\vec{h}(h_x, h_y, h_z)$ denote respectively, the perturbations in density $\rho$, pressure $p$, velocity $(0,0,0)$ and magnetic field $\vec{H}(H_0(z), 0, 0)$. Then the linearized perturbation equations governing the motion of Rivlin-Ericksen fluid and Maxwell's equation are

$$\frac{\rho}{\varepsilon}\frac{\partial\vec{v}}{\partial t} = -\nabla\delta p + \vec{g}\,\delta\rho + \frac{\mu_e}{4\pi}[(\nabla\times\vec{h})\times\vec{H} + (\nabla\times\vec{H})\times\vec{h}] - \frac{\rho}{k_1}\left(\nu + \nu'\frac{\partial}{\partial t}\right)\vec{v}, \quad \text{...(1)}$$

$$\nabla\cdot\vec{v} = 0, \quad \text{...(2)}$$

$$\varepsilon\frac{\partial}{\partial t}(\delta\rho) = -\omega\frac{\partial\rho}{dz}, \quad \text{...(3)}$$

$$\varepsilon\frac{\partial\vec{h}}{\partial t} = (\vec{H}\cdot\nabla)\vec{v} - (\vec{v}\cdot\nabla)\vec{H} \quad \text{...(4)}$$

$$\nabla\cdot\vec{h} = 0. \quad \text{...(5)}$$

Equations (1) and (2) are linearized perturbed equations of motion and continuity whereas equation (3) ensures that the density of every particle remains unchanged as we follow it with its motion. Equations (4) and (5) are linearlized perturbed, Maxwell's equations.

Analysing the disturbance into normal modes, we seek solutions whose dependence on $x$, $y$ and $t$ is given by

$$\exp(ik_x x + ik_y y + nt), \quad \text{...(6)}$$

where $k_x$, $k_y$ are horizontal wave numbers, $k = (k_x^2 + k_y^2)^{1/2}$ and $n$ is in general, a complex constant.

Equations (1)-(5), using expression (6), give

$$\frac{\rho}{\varepsilon}\left[\left(1 + \frac{\varepsilon\nu'}{k_1}\right)n + \frac{\varepsilon\nu}{k_1}\right]u = -ik_x\delta p + \frac{\mu_e}{4\pi}h_z\,dH_0 \quad \text{...(7)}$$

$$\frac{\rho}{\varepsilon}\left[\left(1 + \frac{\varepsilon\nu'}{k_1}\right)n + \frac{\varepsilon\nu}{k_1}\right]v = -ik_y\delta p + \frac{\mu_e}{4\pi}(ik_x h_y - ik_y h_x) \quad \text{...(8)}$$

$$\frac{\rho}{\varepsilon}\left[\left(1+\frac{\varepsilon\nu'}{k_1}\right)n+\frac{\varepsilon\nu}{k_1}\right]w = -D\delta p - \frac{g(D\rho)}{\varepsilon n}w + \frac{\mu_e}{4\pi}\left(ik_xh_z - Dh_xh_x\frac{DH_0}{H_0}\right) \qquad ...(9)$$

$$ik_xu + ik_yv + Dw = 0 \qquad ...(10)$$

$$ik_xh_x + ik_yh_y + Dh_z = 0 \qquad ...(11)$$

$$\varepsilon nh_x = ik_xH_0u - wdH_0 \qquad ...(12)$$

$$\varepsilon nh_y = ik_xH_0v \qquad ...(13)$$

$$\varepsilon nh_z = ik_xH_0w. \qquad ...(14)$$

Eliminating $u$, $v$, $h_x$, $h_y$, $h_z$ and $\delta p$ from equations (7)-(9) and using (10)- (14), after a little algebra, we get

$$D\left[\frac{\rho}{\varepsilon}\left\{\left(1+\frac{\varepsilon\nu'}{k_1}\right)n+\frac{\varepsilon\nu}{k_1}\right\}Dw\right] - k^2\frac{\rho}{\varepsilon}\left[\left(1+\frac{\varepsilon\nu'}{k_1}\right)n+\frac{\varepsilon\nu}{k_1}\right]w$$
$$+\frac{gk^2}{\varepsilon n}(D\rho)w + \frac{\mu_ek_x^2}{4\pi\varepsilon n}\left\{H_0^2(D^2-k^2)w + D(H_0^2)Dw - \frac{H_0^2\,(Dw)(D\varepsilon)}{\varepsilon}\right\} = 0 \qquad ...(15)$$

which, on simplification, can be written as

$$\left[\frac{\rho}{\varepsilon}\left\{\left(1+\frac{\varepsilon\nu'}{k_1}\right)n+\frac{\varepsilon\nu}{k_1}\right\}\right](D^2-k^2)w + \left\{\frac{n}{\varepsilon}(D\rho)\left(1+\frac{\varepsilon\nu'}{k_1}\right)+\frac{n\rho}{\varepsilon^2}(D\varepsilon)\left(1+\frac{\varepsilon\nu'}{k_1}\right)\right.$$
$$\left.+\frac{n\rho}{k_1}D\nu' - \frac{n\rho\nu'}{k_1^2}dk_1 + \frac{\nu}{k_1}D\rho + \frac{\rho}{k_1}D\nu - \frac{\rho\nu}{k_1^2}Dk_1\right\}Dw + \frac{gk^2(D\rho)w}{\varepsilon n}$$
$$+\frac{\mu_ek_x^2}{4\pi\varepsilon n}\left\{H_0^2(D^2-k^2)w + D(H_0^2)Dw - \frac{H_0^2}{\varepsilon}(Dw)(D\varepsilon)\right\} = 0 \qquad ...(16)$$

Equation (16) is the general equation governing the stability of a stratified Rivlin-Ericksen fluid in a stratified porous medium in the presence of a variable horizontal magnetic field.

## 2.2 THE CASE OF EXPONENTIALLY VARYING STRATIFICATIONS

Assume the stratifications in density, kinematic viscosity, kinematic viscoelasticity, medium porosity, medium permeability and magnetic field of the forms

$$\rho = \rho_0e^{\beta z},\ \mu = \mu_0e^{\beta z},\ \mu' = \mu_0'e^{\beta z},\ \varepsilon = \varepsilon_0e^{\beta z},\ k_1 = k_{10}e^{\beta z},\ H_0^2 = H_1^2e^{\beta z} \qquad ...(17)$$

where $\rho_0, \mu_0, \mu_0', \varepsilon_0, k_{10}, H_1$ and $\beta$ are constants. Equations (17) imply that the kinematic viscosity $\nu\left(=\frac{\mu}{\rho}=\frac{\mu_0}{\rho_0}\right)$, the kinematic viscoelasticity $\nu'\left(=\frac{\mu'}{\rho}=\frac{\mu_0'}{\rho_0}\right)$, and the Alfvén velocity $V_A\left(=\sqrt{\frac{\mu_e H_0^2}{4\pi\rho}}=\sqrt{\frac{\mu_e H_1^2}{4\pi\rho_0}}\right)$ are constant everywhere.

Using stratifications of the form (17), equation (16) transforms to

$$\left[n+\frac{\varepsilon_0}{\kappa_{10}}(\nu_0+\nu_0' n)+\frac{k_x^2 V_A^2}{n}\right](D^2-k^2)w+\frac{gk^2\beta}{n}w=0 \quad \text{...(18)}$$

The general solution of equation (18) is

$$w=Ae^{m_1 z}+Be^{m_2 z} \quad \text{...(19)}$$

where $A, B$ are arbitrary constants; $m_1, m_2$ are given by

$$\left.\begin{aligned} m_{1,2} &= \pm k\left(1-\frac{g\beta}{nL}\right)^{1/2} \\ \text{and} \quad L &= \left[n+\frac{\varepsilon_0}{\kappa_{10}}(v_0+nv_0')+\frac{k_x^2 V_A^2}{n}\right] \end{aligned}\right\} \quad \text{...(20)}$$

Here we consider the fluid to be confined between two rigid planes $z=0$ and $z=d$. The boundary conditions for the case of two rigid surfaces are

$$w=0 \text{ at } z=0 \text{ and } z=d. \quad \text{...(21)}$$

The vanishing of $w$ at $z=0$ is satisified by the choice

$$w=A(e^{m_1 z}-e^{m_2 z})$$

while the vanishing of $w$ at $z=d$ requires

$$e^{(m_1-m_2)d}=1$$

i.e.

$$(m_1-m_2)d=2is\pi$$

where $s$ is an integer.

Inserting the values of $m_1$, $m_2$ from (20) in above equation, we obtain

$$1-\frac{g\beta}{nL}=-\frac{s^2\pi^2}{k^2d^2},$$

which on simplification gives

$$\left(1+\frac{\varepsilon_0\nu_0'}{k_{10}}\right)n^2+\frac{\varepsilon_0\nu_0}{k_{10}}n-\frac{g\beta k^2d^2}{k^2d^2+s^2\pi^2}=0. \qquad \text{...(22)}$$

If $\beta < 0$ (stable stratification), equation (22) does not admit any positive root of $n$ and so the system is always stable for disturbance of all wave numbers.

If $\beta > 0$ (unstable stratification), the system is stable or unstable according as

$$k_x^2\,V_A^2 >< gk^2\beta/L \qquad \text{...(23)}$$

The system is clearly unstable in the absence of a magnetic field. However, the system can be completely stabilized by large enough magnetic field as can be seen from equation (23), if

$$V_A^2 > gk^2\beta/Lk_x^2.$$

Thus if $\beta > 0$ and $k_x^2V_A^2 < gk^2B/L$ equation (22) has atleast one positive root. Let $n_0$ denotes the positive root of equation (22). Then

$$\left(1+\frac{\varepsilon\nu'}{k_1}\right)n_0^2+\frac{\nu\varepsilon}{k_1}n_0+\left(k_x^2\,V_A^2-\frac{gk^2\beta}{L}\right)=0 \qquad \text{...(24)}$$

as $$\nu_0=\frac{\mu_0}{\rho_0}=\frac{\mu}{\rho}=\nu,\ \frac{\varepsilon_0}{\kappa_{10}}=\frac{\varepsilon}{\kappa_1},\ \nu_0'=\frac{\mu_0'}{\rho_0}=\frac{\mu'}{\rho}=\nu'$$

and $$V_A^2=\frac{\mu_e H_1^2}{4\pi\rho_0}=\frac{\mu_e H_0^2}{4\pi\rho}.$$

To find the role of kinematic viscosity, kinematic viscoelasticity, medium permeablity and magnetic field on the growth rate of unstable modes, we examine the natures of

$$\frac{dn_0}{dv},\ \frac{dn_0}{dv'},\ \frac{dn_0}{dk_1} \text{ and } \frac{dn_0}{dV_A}.$$

Equation (24) yields

$$\frac{dn_0}{d\nu} = -\frac{\varepsilon n_0}{2n_0\,(k_1 + \varepsilon v') + \varepsilon\nu}, \qquad \ldots(25)$$

$$\frac{dn_0}{d\nu'} = -\frac{\varepsilon n_0^2}{2n_0\,(k_1 + \varepsilon\nu')\,\varepsilon\nu}, \qquad \ldots(26)$$

$$\frac{dn_0}{dk_1} = \frac{\varepsilon n_0\,(\nu + n\nu')}{k_1\,[2n_0\,(k_1 + \varepsilon\nu') + \varepsilon\nu]}, \qquad \ldots(27)$$

$$\frac{dn_0}{dV_A} = -\frac{2k_x^2 V_A}{2n_0\,(k_1 + \varepsilon\nu') + \varepsilon\nu}. \qquad \ldots(28)$$

It is clear from equation (25)-(28) that the growth rates decrease with the increase in kinematic viscosity, kinematic viscoelasticity and magnetic field whereas the growth rates increase with the increase in medium permeability.

## 3. EFFECT OF SUSPENDED PARTICLES

### 3.1 PERTURBATION EQUATIONS

The initial stationary state whose stability we wish to examine is that of an incompressible Rivlin-Ericksen fluid of variable density, kinematic viscosity and kinematic viscoelasticity arranged in horizontal strata embedded by suspended particles (fine dust) in porous medium of variable porosity and permeability. Consider an infinite horizontal fluid-particle layer of thickness $d$ bounded by the planes $z = 0$ and $z = d$. The relevant equations of motion for the fluid are given by

$$\frac{\rho}{\varepsilon}\left[\frac{\partial\vec{V}}{\partial t} + \frac{1}{\varepsilon}(\vec{V}\,.\,\nabla)\vec{V}\right] = -\nabla p + \rho\vec{g} - \frac{\rho}{k_1}\left(\nu + \nu'\frac{\partial}{\partial t}\right)\vec{V} + \frac{kN}{\varepsilon}(\vec{u} - \vec{v}) \qquad \ldots(29)$$

$$\nabla\,.\,\vec{v} = 0, \qquad \ldots(30)$$

$$\varepsilon\frac{\partial\rho}{\partial t} + (\vec{v}\cdot\nabla)\rho = 0. \qquad \ldots(31)$$

Here $\vec{u}(\bar{x}, t)$ and $N(\bar{x}, t)$ denote the velocity and number density of the particles respectively, $k = 6\pi\rho\nu\eta$, where $\eta$ is particle radius, is the Stokes drag coefficient, $\vec{u} = (l, r, s)$ and $\bar{x} = (x, y, z)$.

If $mN$ is the mass of particles per unit volume, then the equations of motion and continuity for the particles are

$$mN\left[\frac{\partial\vec{u}}{\partial t} + \frac{1}{\varepsilon}(\vec{u}\cdot\nabla)\,\vec{u}\right] = kN\,(\vec{v} - \vec{u}) \qquad \ldots(32)$$

$$\varepsilon \frac{\partial N}{\partial t} + \nabla \cdot (N\vec{u}) = 0. \qquad \text{...(33)}$$

The presence of particles adds an extra force term proportional to the velocity difference between particles and fluid and appears in the equation of motion (30). Since the force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid, there must be an extra force term, equal in magnitude but opposite in sign, in the equations of motion for the particles (32). The buoyancy force on the particles is neglected. Inerparticle reactions are not considered either since we assume that the distances between particles are quite large compared with their diameters. These assumptions have been used in writing the equations of motion for the particles (32).

Let $\delta\rho$, $\delta p$ and $\vec{v}(u, v, w)$ denote, respectively the perturbations in fluid density $\rho$, pressure $p$ and velocity (0,0,0). Then the linearized perturbation equations of the fluid-particle layer become

$$\frac{\rho}{\varepsilon}\frac{\partial \vec{v}}{\partial t} = -\nabla\,\delta p + \vec{g}(\delta p) - \frac{\rho}{k_1}\left(\nu + \nu'\frac{\partial}{\partial t}\right)\vec{v} + \frac{kN}{\varepsilon}(\vec{u} - \vec{v}) \qquad \text{...(34)}$$

$$\left(\frac{m}{k}\frac{\partial}{\partial t} + 1\right)\vec{u} = \vec{v} \qquad \text{...(35)}$$

together with equations (2) and (3).

Eliminating $\vec{u}$ from equation (34) by making use of (35) and using expression (6), equations (34), (2) and (3) yield

$$\frac{\rho}{\varepsilon}\left[n + \frac{\varepsilon}{\kappa_1}(\nu + n\nu') + \frac{mNn/\rho}{\frac{mn}{K} + 1}\right]u = -ik_x\delta p \qquad \text{...(36)}$$

$$\frac{\rho}{\varepsilon}\left[n + \frac{\varepsilon}{\kappa_1}(\nu + n\nu') + \frac{mNn/\rho}{\frac{mn}{k} + 1}\right]v = -ik_y\delta p \qquad \text{...(37)}$$

$$\frac{\rho}{\varepsilon}\left[n + \frac{\varepsilon}{\kappa_1}(\nu + n\nu') + \frac{mNn/\rho}{\frac{mn}{k} + 1}\right]w = -D(\delta p) - g(\delta\rho) \qquad \text{...(38)}$$

$$ik_x u + ik_y v + Dw = 0 \qquad \text{...(39)}$$

$$\varepsilon n\delta\rho = -wD\rho. \qquad \text{...(40)}$$

Eliminating $u$, $v$ and $\delta p$ from equations (36)-(38) and using (39) and (40), we obtain

$$\left[ n + \frac{\varepsilon}{\kappa_1}(\nu + n\nu') + \frac{mNn/\rho}{\frac{mn}{k} + 1} \right](D^2 - k^2)w + \left[ \frac{1}{k_1} D\{\varepsilon(\nu + n\nu')\} \right.$$

$$\left. - \frac{\varepsilon\,(\nu + n\nu')Dk_1}{k_1^2} + \frac{mn}{(mn/k + 1)} \left\{ N\left( -\frac{1}{\rho^2} \right) D\rho + \frac{1}{\rho} DN \right\} \right] Dw + \frac{gk^2(d\rho)w}{\rho n} = 0 \qquad \text{...(41)}$$

### 3.2 THE CASE OF EXPONENTIALLY VARYING STRATIFICATIONS

Assume the stratifications in fluid density, fluid viscosity, fluid viscoelasticity, suspended particles number density, medium porosity and medium permeability of the forms

$$\rho = \rho_0 e^{\beta z},\ \mu = \mu_0 e^{\beta z},\ \mu_0' = \mu_0' e^{\beta z},\ N = N_0 e^{\beta z},\ \varepsilon = \varepsilon_0 e^{\beta z},\ k_1 = k_{10} e^{\beta z} \qquad \text{...(42)}$$

where $\rho_0$, $\mu_0$, $\mu_0'$, $N_0$, $\varepsilon_0$, $k_{10}$ and $\beta$ are constants and so the kinematic viscosity $\nu = \frac{\mu}{\rho} = \frac{\mu_0}{\rho_0} (= \nu_0)$ and the kinematic viscoelasticity $\nu' = \frac{\mu'}{\rho} = \frac{\mu_0'}{\rho_0} (= \nu_0')$ are contrants everywhere.

Using stratifications of the form (42), equation (41) transfoms to

$$\left[ n + \frac{\varepsilon_0}{\kappa_{10}}(\nu_0 + \nu_0' n) + \frac{mN_0 n/\rho_0}{\left( \frac{mn}{k} + 1 \right)} \right](D^2 - k^2)w + \frac{gk^2\beta}{n} w = 0 \qquad \text{...(43)}$$

The general solution of equation (43) is

$$w = Ae^{q_1 z} + Be^{q_2 z} \qquad \text{...(44)}$$

where $A$, $B$ are constants; $q_1$, $q_2$ are given by

$$q_{1,2} = \pm k \left[ 1 - \frac{g\beta}{nL_2} \right]^{1/2} \qquad \text{...(45)}$$

and

$$L_2 = \left[ n + \frac{\varepsilon_0}{\kappa_{10}}(\nu_0 + \nu_0' n) + \frac{mN_0 n/\rho_0}{\left( \frac{mn}{k} + 1 \right)} \right] \qquad \text{...(46)}$$

Here also we consider the fluid to be confined between two rigid planes at $z = 0$ and $z = d$. The boundary conditions are

$$w = 0 \text{ at } z = 0 \text{ and } z = d \qquad \text{...(47)}$$

The vanishing of $w$ at $z = 0$ is satisfied by the choice

$$\omega = A\,(e^{q_1 z} - e^{q_2 z}),$$

while the vanishing of $w$ at $z = d$ requires

$$e^{(q_1 - q_2)d} = 1$$

i.e.
$$(q_1 - q_2)d = 2is\pi$$

where $s$ is an integer.

Inserting the values of $q_1$, $q_2$ form (45) in above equation, we obtain

$$1 - \frac{g\beta}{nL_2} = -\frac{s^2\pi^2}{d^2},$$

which on simplification yields

$$\left(1 + \frac{\varepsilon_0 \nu_0'}{k_{10}}\right) n^3 + \left[\frac{k}{m}\left(1 + \frac{\varepsilon_0}{k_{10}}\nu_0'\right) + \frac{\varepsilon_0 \nu_0}{k_{10}} + \frac{kN_0}{\rho_0}\right] n^2 + \left[\frac{\nu_0 \varepsilon_0}{k_{10}}\frac{k}{m} - \frac{g\beta k^2 d^2}{k^2 d^2 + s^2\pi^2}\right] n - \frac{g\beta k^2 d^2}{k^2 d^2 + s^2 \pi^2}\frac{k}{m} = 0. \qquad \text{...(48)}$$

If $\beta < 0$ (stable stratification), equation (48) does not allow any positive root of $n$ and so the system is always stable for disturbances of all wave numbers.

If $\beta > 0$ (unstable stratification), equation (48) has one positive root and so the system is unstable for disturbances of all wave numbers. Let $n_0$ denote the positive root of equation (48). Then

$$\left[1 + \frac{\varepsilon \upsilon'}{k_1}\right] n_0^3 + \left[\frac{k}{m} + \frac{\varepsilon \upsilon}{k_1} + \frac{k}{m} + \frac{\varepsilon \upsilon'}{k_1} + \frac{kN}{\rho}\right] n_0^2 + \left[\frac{\upsilon \varepsilon}{k_1}\frac{k}{m} - \frac{g\beta k^2 d^2}{k^2 d^2 + s^2 \pi^2}\right] n_0 - \frac{g\beta k^2 d^2}{k^2 d^2 + s^2\pi^2}\frac{k}{m} = 0 \qquad \text{...(49)}$$

as $\nu_0 = \dfrac{\mu_0}{\rho_0} = \dfrac{\mu}{\rho} = \nu,\ \nu_0' = \dfrac{\mu_0'}{\rho_0} = \dfrac{\mu'}{\rho} = \nu',\ \dfrac{N_0}{\rho_0} = \dfrac{N}{\rho}$

and $\dfrac{\varepsilon_0}{k_{10}} = \dfrac{\varepsilon}{k_1}.$

To find the roles of fluid viscosity, fluid viscoelasticity, suspended particles number density and medium permeability on the growth rates of unstable modes, we examine the natures of $\dfrac{dn_0}{d\nu}, \dfrac{dn_0}{d\nu'},$ $\dfrac{dn_0}{dN}$ and $\dfrac{dn_0}{dk_1}$ analytically. Equation (49) yields

$$\frac{dn_0}{d\nu} = -\frac{n_0\varepsilon\left(n_0 + \dfrac{k}{m}\right)}{k_1\left[3n_0^2\left(1+\dfrac{\varepsilon\nu'}{k}\right) + 2n_0\left(\dfrac{k}{m} + \dfrac{\varepsilon\nu}{k_1} + \dfrac{k\varepsilon}{mk_1}\nu' + \dfrac{kN}{\rho}\right) + \dfrac{\nu\varepsilon}{k_1}\dfrac{k}{m} - \dfrac{g\beta k^2 d^2}{k^2 d^2 + s^2\pi^2}\right]} \quad \text{...(50)}$$

$$\frac{dn_0}{d\nu'} = -\frac{n_0^2\varepsilon\left(n_0 + \dfrac{k}{m}\right)}{k_1\left[3n_0^2\left(1+\dfrac{\varepsilon\nu'}{k_1}\right) + 2n_0\left(\dfrac{k}{m} + \dfrac{\varepsilon\nu}{k_1} + \dfrac{k\varepsilon}{mk_1}\nu' + \dfrac{kN}{\rho}\right) + \dfrac{\nu\varepsilon}{k_1}\dfrac{k}{m} - \dfrac{g\beta k^2 d^2}{k^2 d^2 + s^2\pi^2}\right]} \quad \text{...(51)}$$

$$\frac{dn_0}{dN} = \frac{n_0\, k/\rho}{3n_0^2\left(1+\dfrac{\varepsilon\nu'}{k_1}\right) + 2\left(\dfrac{k}{m} + \dfrac{\varepsilon\nu}{k_1} + \dfrac{k\varepsilon}{mk_1}\nu' + \dfrac{kN}{\rho}\right)} \quad \text{...(52)}$$

and

$$\frac{dn_0}{dk_1} = \frac{n_0\varepsilon\left[n_0^2\nu' + n_0\left(\nu + \dfrac{k}{m}\nu'\right) + \nu\dfrac{k}{m}\right]}{k_1^2\left[3n_0^2\left(1+\dfrac{\varepsilon\nu'}{k_1}\right) + 2n_0\left(\dfrac{k}{m} + \dfrac{\varepsilon\nu}{k_1} + \dfrac{k}{m}\dfrac{\varepsilon\nu'}{k_1} + \dfrac{kN}{\rho}\right) + \dfrac{\nu\varepsilon}{k_1}\dfrac{k}{m} - \dfrac{g\beta k^2 d^2}{k^2 d^2 + s^2\pi^2}\right]} \quad \text{...(53)}$$

It is clear from equations (50) and (51) that the growth rates decrease or increase with the increase in kinematic viscosity and kinematic viscoelasticity according to

$$3n_0^2\left(1+\frac{\varepsilon\nu'}{k_1}\right) + 2n_0\left(\frac{k}{m} + \frac{\varepsilon\nu}{k_1} + \frac{k\varepsilon\nu'}{mk_1} + \frac{kN}{\rho}\right) + \frac{\nu\varepsilon}{k_1}\frac{k}{m} >< \frac{g\beta^2 d^2}{k^2 d^2 + s^2\pi^2} \quad \text{...(54)}$$

**Equation (52) implies that the growth rates increase with the increase in the suspended particles number density. However, the growth rates increase or decrease with the increase in medium permeability according to (54) as is evident from equation (53).**

## REFERENCES

- [1] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Dove Publications, New York, 1981.
- [2] J.W. Scanlon and L.A. Segel, *Phys. Fluids* **16** (1973), 1573.
- [3] V.I. Palaniswamy and C.M. Purushotham, *Phys. Fluids.* **24** (1981), 1224.
- [4] A. Kent, *Phys. Fluids*, **9** (1966), 1286.
- [5] R.C. Sharma, *Acta Phys. Hungar;* **38** (1975), 293.
- [6] J.G. Oldroyd, *Proc. Roy. Soc. London Ser. A.* **245** (1958), 278.
- [7] A. Garg, R.K. Srivastava *and K.K. Singh, Proc. Nat. Acad. Sci. India Sect. A* **64** (1994), 355.

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# INSTABILITY OF TWO ROTATING VISCOELASTIC SUPERPOSED FLUIDS WITH SUSPENDED PARTICLES IN POROUS MEDIUM

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## ABSTRACT

The instability of the plane interface between two Oldroydian viscoelastic superposed fluids permeated with suspended particles and uniform rotation in porous medium is considered. The cases of two uniform viscoelastic fluids separated by a horizontal boundary and exponentially varying density, viscosity and suspended particles number density are considered. The system is found to be stable for potentially stable configuration/stable stratification and unstable for potentially unstable configuration/unstable stratification. The viscoelasticity, suspended particles, rotation and porous medium effects do not affect the stability or instability of the system.

**1. INTRODUCTION.** A detailed account of the instability of the plane interface between two Newtonian viscous fluids under varying assumptions of hydrodynamics and hydromagnetics, has been given by Chandrasekhar[1], Bhatia and Steiner[2] have studied the problem of thermal instability of a Maxiwellian viscoelastic fluid in the presence of rotation and have found that the rotation has a destablizing effect in contrast to the stabilizing effect on Newtonian fluid. Bhatia and Steiner[3] have also considered the problem of thermal instability of a Maxwellian fluid in hydromagnetics and have found that the magnetic field has stabilizing effect on viscoelastic fluid just as in the case of Newtonian fluid. Eltayeb[4] has studied the convective instability in a rapidly rotating Oldroydian viscoelastic fluid. An experimental demonstration by Toms and Strawbridge[5] reveals that a dilute solution of methyl methacrylate in *n*-butyl acetate agrees well with the theoretical model of oldroyd fluid.

In geophysical situations, more often than not, the fluid is not pure, but may instead be permeated with suspended (or dust) particles. Şcanlon and Segel[6] have considered the effect of suspended particles on the onset of Bénard convection and found that the critical Rayleigh number was reduced solely because the heat capacity of the pure gas was supplemented by that of the particles. The effect of suspended particles was thus found to destablize the layer. The medium has been considered to be non- porous in all the above studies.

The flow through porous medium has been of considerable interest in recent years particularly among petroleum engineers and geophysical fluid dynamicists. A macroscopic equation which describes incompressible flow of Newtonian fluid of viscosity $\mu$ through a homogeneous and isotropic porous medium of permeability $k_1$ is the Darcy's equation. As a result of this macroscopic law, the usual viscous term in the equations of fluid motion is replaced by the resistance term $-(\mu/k_1)\,\vec{q}$, where $\vec{q}$ is the filter velocity of the fluid. The thermal instability of fluids in a porous medium in the presence of suspended particles has been studied by Sharma and Sharma[7]. The effects of suspended particles and medium permeability were found to destabilize the layer.

The present paper attempts to study the stability of the plane interface separating two incompressible superposed rotating, Oldroydian viscoelastic fluids, permeated with suspended particles, in a porous medium. The constitutive relations of the type of Oldroyd fluid were proposed and studied by Oldryod[8]. The knowledge regarding fluid-particle mixtures is not commensurate with thier scientific and industrial importance. The analysis would be relevant to the stability of some polymer solutions like a dilute solution of methyl methacrylate in $n$-butyl acetate and to the stability of some Maxwellian viscoelastic fluids and the problem finds its usefulness in several geophysical situations and in chemical technology. These aspects form the motivation for the present study.

**2. PERTURBATION EQUATIONS.** Let $T_{ij}, z_{ij}, e_{ij}, \delta_{ij}, \mu, \lambda, \lambda_0(<\lambda)$, $p, v_i, x_i$ and $d/dt$ denote respectively the total stress tensor, the shear stress tensor, the rate-of-strain tensor, the Kronecker delta, the viscosity, the stress relaxation time, the strain retardation time, the isotropic pressure, the velocity vector, the position vector and the mobile operator. Then the Oldroydian viscoelastic fluid is described by the constitutive relations

$$T_{ij} = -p\delta_{ij} + z_{ij},$$

$$(i + \lambda \frac{d}{dt})\, z_{ij} = 2\mu\,(1 + \lambda_0 \frac{d}{dt})\, e_{ij}, \qquad \ldots(2.1)$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

Relations of the type (2.1) were posed and studied by Oldroyd[8]. Oldroyd[8] also showed that many rheology equations of state of general validity, reduce to (2.1) when linearized. When $\lambda_0 = 0$ the fluid is termed as Maxwellian viscoelastic whereas $\lambda = \lambda_0 = 0$ yields the fluid to be Newtonian viscous fluid.

Consider a static state in which an incompressible Oldroydian viscoelastic fluid permeated with suspended partiles is arranged in horizontal strata in porous medium and the pressure $p$ and density $\rho$ are functions of the vertical coordinate $z$ only. A uniform rotation

$\vec{\Omega}(0, 0, \Omega)$ pervades the whole system. The character of the equilibrium of this initial static state is determined, as usual by supposing that the system is slightly disturbed and then by following its further evolution.

Let $\rho$, $\mu$, $p$, $\boldsymbol{u}$ $(u, v, w)$ and $\vec{\Omega}$ $(0, 0, \Omega)$ denote respectively, the density, the viscosity, the pressure, the velocity of pure fluid and the uniform rotation; $\boldsymbol{v}$ $(\bar{x}, t)$ and $N$ $(\bar{x}, t)$ denote the velocity and number density of the particles respectively. $k = \sigma\pi\rho\upsilon\eta$ where $\eta$ is the particle radius is the Stokes' drag coefficient, $\boldsymbol{v} = (l, r, s)$, $\bar{x} = (x, y, z)$ and $\vec{\lambda}' = (0, 0, 1)$. Let $\in$, $k_1$ and $g$ stand for medium porosity, medium permeability and acceleration due to gravity respectivley. Then the equations of motion and continuity for the rotating Oldroydian viscoelastic fluid permeated with suspended particles in a porous medium are

$$\frac{\rho}{\varepsilon}\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{\partial \boldsymbol{u}}{\partial t}+\frac{1}{\varepsilon}(\boldsymbol{u}.\nabla)\,\boldsymbol{u}\right]=\left(1+\lambda\frac{\partial}{\partial t}\right)[-\nabla(p-\frac{\rho}{2}|\vec{\Omega}\times\boldsymbol{x}|^2)$$

$$-\rho g\vec{\lambda}+\frac{kN}{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})+\frac{2\rho}{\varepsilon}(\boldsymbol{u}\times\vec{\Omega})]-(1+\lambda_0\frac{\partial}{\partial t})\frac{\mu}{k_1}\boldsymbol{u} \qquad \text{...(2.2)}$$

$$\nabla.\boldsymbol{u}=0 \qquad \text{...(2.3)}$$

Since the density of a fluid particle moving with the fluid remains unchanged, we have

$$\varepsilon\frac{\partial\rho}{\partial t}+(\boldsymbol{u}\,.\,\nabla)\rho=0. \qquad \text{...(2.4)}$$

In the equations of motion (2.2), by assuming a uniform particle size, spherical shape and small relative velocities between the fluid and particles, the presence of particles adds an extra force term proportional to the velocity differences between the particles and the fluid. Since the force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid, there must be an extra force term, equal in magnitude but opposite in sign, in the equations of motion of the particles. The distances between particles are assumed quite large compared with their diameter so that interparticle reactions are ignored. The effects of pressure, gravity and Darcian force on the suspended particles are negligibly small and therefore ignored. If $mN$ is the mass of particles per unit volume, then the equations of motion and continuity for the particles, under the above assumptions, are

$$mN\left[\frac{\partial \boldsymbol{v}}{\partial t}+\frac{1}{\varepsilon}(\boldsymbol{v}\,.\,\nabla)\boldsymbol{v}\right]=kN(\boldsymbol{u}-\boldsymbol{v}), \qquad \text{...(2.5)}$$

$$\varepsilon\frac{\partial N}{\partial t}+\nabla\,.\,(N\boldsymbol{v})=0. \qquad \text{...(2.6)}$$

Let $\delta\rho$, $\delta p$, $\boldsymbol{u}(u, v, w)$ and $\boldsymbol{v}$ $(l, r, s)$ denote respectively, the perturbations in fluid density $\rho$, fluid pressure $p$, fluid velocity (0, 0,

0) and particle velocity (0, 0, 0). Then the linearized perturbation equations of the fluid-particle layer are

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\rho}{\varepsilon}\frac{\partial \boldsymbol{u}}{\partial t}=\left(1+\lambda\frac{\partial}{\partial t}\right)\left[-\nabla\delta p+\boldsymbol{g}\delta\rho+\frac{kN}{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})+\frac{2\rho}{\varepsilon}(\boldsymbol{u}+\vec{\Omega})\right]-\left(1+\lambda_0\frac{\partial}{\partial t}\right)\frac{\mu}{k_1}\boldsymbol{u} \qquad \text{...(2.7)}$$

$$\nabla\,.\,\boldsymbol{u}=0, \qquad \text{...(2.8)}$$

$$\varepsilon\frac{\partial}{\partial t}\delta\rho=-w\frac{d\rho}{dz}, \qquad \text{...(2.9)}$$

$$\left(\frac{m}{k}\frac{\partial}{\partial t}+1\right)\boldsymbol{v}=\boldsymbol{u}. \qquad \text{...(2.10)}$$

Analyzing the disturbances into normal modes, we seek solutions whose dependence on $x$, $y$ and $t$ is given by

$$\exp\left(ik_x x+ik_y y+nt\right), \qquad \text{...(2.11)}$$

where $k_x$, $k_y$ are wave numbers along $x$ and $y$-directions, $k^2=k_x^2+k_y^2$ and $n$ is in general, a complex constants.

For perturbations of the form (2.11), writing component equations and eliminating $u$, $v$, $\delta\rho$ and $\delta p$ Eqs. (2.7)-(2.10) yield

$$\left[(1+\lambda n)(zn+1)n+(1+\lambda_0 n)(zn+1)\frac{\varepsilon v}{k_1}\right]\left[D(\rho Dw)-k^2\rho w\right]+n(1+\lambda n)\left[D(mNDw)-mNk^2w\right)+$$

$$+\left[gk^2\frac{(1+\lambda n)(zn+1)}{n}\right](D\rho)w+4(1+\lambda n)^2(zn+1)^2\Omega^2$$

$$\left[D\left\{\frac{\rho Dw}{(1+\lambda n)(zn+1)n+(1+\lambda n)\dfrac{mnN}{\rho}+(1+\lambda_0 n)(zn+1)\dfrac{v\varepsilon}{k_1}}\right\}\right]=0 \qquad \text{...(2.12)}$$

where $v\ (=\mu/\rho)$ stands for kinematic viscosity.

## 3. TWO UNIFORM VISCOELASTIC FLUIDS SEPARATED BY A HORIZONTAL BOUNDARY

Here we consider the case when two superposed uniform Oldroydian viscoelastic fluids of densities $\rho_1$ and $\rho_2$ uniform viscosities $\mu_1$ and $\mu_2$ uniform suspended particles densities $N_1$ and $N_2$ are separated by a horizontal boundary at $z=0$. The subscripts 1 and 2 distinguish the lower and the upper fluids respectively. Then, in each region of constant $\rho$, contant $\mu$ and constant $mN$, Eq. (2.12) reduces to

$$(D^2 - \chi^2)w = 0, \qquad \text{...(3.1)}$$

where

$$\chi = \frac{k}{\left[1 + \dfrac{4\Omega^2 (1+\lambda n)^2 (zn+1)^2}{\left[n(1+\lambda n)(zn+1)n(1+\lambda n)\dfrac{mN}{\rho} + (1+\lambda_0 n)(zn+1)\dfrac{\upsilon\varepsilon}{k_1}\right]^2}\right]^{1/2}}$$

and in case of highly viscous fluid

$$\chi = \frac{k}{\left[1 + \dfrac{1}{2}\dfrac{4\Omega^2 (1+\lambda n)^2 (zn+1)^2}{\left[n(1+\lambda n)(zn+1) + n(1+\lambda n)\dfrac{mN}{\rho} + (1+\lambda_0 n)(zn+1)\dfrac{\upsilon\varepsilon}{k_1}\right]^2}\right]} \qquad \text{...(3.2)}$$

The general solution of Eq. (3.1) is

$$w = Ae^{+\chi z} + be^{-\chi z} \qquad \text{...(3.3)}$$

where $A$ and $B$ are arbitrary constants. The boundary conditions to be statisfied in the present problem are

(*i*) The velocity $w$ should vanish when $z \to +\infty$ (for the upper fluid) and $z \to -\infty$ (for the lower fluid).

(*ii*) $w(z)$ is continuous at $z = 0$.

(*iii*) The pressure should be continuous across the interface.

Applying the boundary conditions (*i*) and (*ii*), we have

$$w_1 = Ae^{+\chi z} \ (z < 0), \qquad \text{...(3.4)}$$

$$w_2 = Ae^{-\chi z} \ (z > 0), \qquad \text{...(3.5)}$$

where the same constant $A$ has been chosen to ensure the continuity of $w$ at $z = 0$.

Here we assume the kinematic viscosities of both fluids to be equal i.e. $\upsilon_1 = \upsilon_2 = \nu$ (Chandrasekhar[1], p.443) and $\dfrac{mN}{\rho} = \dfrac{mN_1}{\rho_1} = \dfrac{mN_2}{\rho_2} (= M)$ as these simplifying assumptions do not obscure any of the essential features of the problem. Integrating Eq. (2.12) across the interface, $z = 0$ we obtain the boundary condition

$$\left[(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\frac{\upsilon\varepsilon}{k_1}\right]\Delta_0\,(\rho Dw)$$

$$+ n(1+\lambda n)\,\Delta_0(mNDw) + (1+n\lambda)(zn+1)\,\frac{gk^2}{n}\,\Delta_0(\rho)w_0$$

$$+ \frac{4(1+\lambda n)^2(zn+1)^2\,\Omega^2}{\left[(1+\lambda n)(zn+1)n + (1+\lambda n)\dfrac{mnN}{\rho} + (1+\lambda_0 n)(zn+1)\dfrac{\varepsilon\upsilon}{k_1}\right]}\,\Delta_0(\rho Dw)$$

$$= 0 \qquad \text{...(3.6)}$$

where $w_0$ is the common value of $w$ at $z = 0$.

Applying the boundary condition (3.6) to the solutions (3.4) and (3.5), we obtain

$$1 + \frac{mn\,(1+\lambda n)\,(N_1+N_2)}{(\rho_1+\rho_2)\left[(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\,\dfrac{\varepsilon\upsilon}{k_1}\right]}$$

$$- \frac{(1+\lambda_n)(zn+1)gk^2(\alpha_2-\alpha_1)}{n\chi\left[(1+\lambda)(zn+1)n + (1+\lambda_0)(zn+1)\dfrac{\varepsilon\upsilon}{k_1}\right]}$$

$$+ \frac{4(1+\lambda n)^2\,(zn+1)^2\,\Omega^2\,(\alpha_2+\alpha_1)}{\left[(1+\lambda n)(zn+1)n + (1+\lambda n)Mn + (1+\lambda_0 n)(zn+1)\dfrac{\varepsilon\upsilon}{k_1}\right]\left[(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\dfrac{\varepsilon\upsilon}{k_1}\right]}$$

$$= 0, \qquad \text{...(3.7)}$$

where
$$\alpha_{1,\,2} = \frac{\rho_{1,\,2}}{\rho_1+\rho_2}.$$

Equation (3.7), after substituting the value of $\chi$ from (3.2) and simplification, yields

$$A_{13}n^{13} + A_{12}n^{12} + A_{11}n^{11} + \ldots + A_2n^2 + A_1n + A_0 = 0, \qquad \text{...(3.8)}$$

where
$$A_{13} = \lambda^4 z^4,$$

$$A_0 = -\,gk(\alpha_2-\alpha_1)\,\frac{\varepsilon\upsilon}{k_1}\left[\frac{\varepsilon^2\upsilon^2}{k_1^2} + 2\Omega^2\right] \qquad \text{...(3.9)}$$

and the coefficients $A_1 - A_{12}$ being quite lengthy and not needed in the discussion of stability, have not been written here.

For the potentially stable arrangement ($\alpha_2 < \alpha_1$) all the coefficients of Eq. (3.8) are positive. So, all the roots of Eq. (3.8) are either real and negative or there are complex roots (which occur in pairs) with negative real parts and the rest negative real roots.

The system is therefore stable in each case.

For the potentially unstable arrangement $(\alpha_2 > \alpha_1)$, the constant term is negative and so there is atleast one change of sign in Eq. (3.8). Therefore, Eq. (3.8) allows atleast one positive root of $n$ meaning thereby instability of the system.

**4. SPECIAL CASE.** In the absence of rotation, Eq. (3.7) after some algebraic simplifications, gives

$$A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0 = 0, \qquad \ldots(4.1)$$

where $A_4 = (\rho_2 + \rho_1)\lambda z,$

$$A_3 = \left[\ (\rho_2 + \rho_1)(z + \lambda + \lambda_0 z \frac{\varepsilon v}{k_1}) + m\lambda(N_1 + N_2)\ \right]$$

$$A_2 = \left[(\rho_2 + \rho_1)(1 + \frac{zv\varepsilon}{k_1} + \lambda_0 \frac{\varepsilon v}{k_1}) + m(N_1 + N_2) - (\rho_1 + \rho_2)\lambda z g k(\alpha_2 - \alpha_1)\right]$$

$$A_1 = \left[\ (\rho_2 + \rho_1)\left\{\ \frac{\varepsilon v}{k_1} - (\alpha_2 - \alpha_1) g k (z + \lambda)\ \right\}\right]$$

and

$$A_0 = -\,(\rho_2 + \rho_1) g k (\alpha_2 - \alpha_1). \qquad \ldots(4.2)$$

For the potentially stable arrangement $(\alpha_2 < \alpha_1)$ all the coefficients of Eq. (4.1) are positive. So, all the roots of Eq. (4.1) are either real and negative or there are complex roots (which occur in pairs) with negative real parts and the rest negative real roots. The system is, therefore, stable in each case.

For the potentially unstable arrangement $(\alpha_2 > \alpha_1)$ the constant term is negative and so there is atleast one change of sign in Eq. (4.1). Equation (4.1) allows atleast one positive root of $n$ thereby meaning instability of the system.

## 5. THE CASE OF EXPONENTIALLY VARYING DENSITY, VISCOSITY AND SUSPENDED PARTICLES NUMBER DENSITY

Let us assume the stratifications in density, viscosity and suspended particles number density of the form

$$\rho = \rho_0 e^{\beta z},\ \mu = \mu_0 e^{\beta z},\ n = N_0 e^{\beta z} \qquad \ldots(5.1)$$

where $\rho_0$, $\mu_0$, $N_0$ and $\beta$ are constants. Equations (5.1) imply that the coefficient of kinematic viscosity $\nu$ is constant everywhere.

Using the stratification of the form (5.1), Eq. (2.12) transforms to

$$\left[\ (1 + \lambda n)(zn + 1)n + (1 + \lambda_0 n)(zn + 1)\,\frac{\varepsilon v_0}{k_1} + M_0 n(1 + \lambda_n)\right.$$

$$+ \frac{4\Omega^2(1+\lambda n)^2\,(zn+1)^2}{(1+\lambda n)(zn+1)n + (1+\lambda n)M_0 n + (1+\lambda_0 n)(zn+1)\,\dfrac{\varepsilon v_0}{k_1}}\Bigg] D^2 w$$

$$+ \Bigg[\,(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\,\frac{\varepsilon v_0}{k_1} + M_0 n(1+\lambda n)$$

$$+ \frac{4\Omega^2(1+\lambda n)^2(zn+1)^2}{(1+\lambda n)(zn+1)n + (1+\lambda n)M_0 n + (1+\lambda_0 n)(zn+1)\,\dfrac{\varepsilon v_0}{k_1}}\Bigg]\beta Dw$$

$$- \Bigg[\,(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\frac{\varepsilon v_0}{k_1} + M_0 n(1+\lambda n)$$

$$- \frac{g\beta(1+\lambda n)(zn+1)}{n}\Bigg] k^2 w = 0 \qquad \text{...(5.2)}$$

The general solution of Eq. (5.2) is

$$w = A_1 e^{q_1 z} + A_2 e^{q_2 z}, \qquad \text{...(5.3)}$$

where $A_1, A_2$ are two arbitrary constant and $q_1, q_2$ are the roots of the equation

$$\Bigg[\,(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\frac{\varepsilon v_0}{k_1} + M_0 n(1+\lambda n)$$

$$+ \frac{4\Omega^2\,(1+\lambda n)^2(zn+1)^2}{(1+\lambda n)(zn+1)n + (1+\lambda n)M_0 n + (1+\lambda_0 n)(zn+1)\dfrac{\varepsilon v_0}{k_1}}\Bigg] q^2$$

$$+ \Bigg[\,(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\frac{\varepsilon n_0}{k_1} + M_0 n(1+\lambda n)$$

$$+ \frac{4\Omega^2\,(1+\lambda n)^2(zn+1)^2}{(1+\lambda n)(zn+1)n + (1+\lambda n)M_0 n + (1+\lambda_0 n)(zn+1)\dfrac{\varepsilon v_0}{k_1}}\Bigg]\beta q$$

$$- \Bigg[\,(1+\lambda n)(zn+1)n + (1+\lambda n)(zn+1)\frac{\varepsilon v_0}{k_1} + M_0 n(1+\lambda n)$$

$$- \frac{g\beta(1+\lambda n)(zn+1)}{n}\Bigg] k^2 = 0. \qquad \text{...(5.4)}$$

If the fluid is susposed to be confined between two rigid planes at $z = 0$ and $z = d$, then the vanishing of $w$ at $z = 0$ is statisfied by the choice

$$w = A(e^{q_1 z} - e^{q_2 z}), \qquad ...(5.5)$$

while the vanishing of $w$ at $z = d$ requires

$$\exp (q_1 - q_2)d = 1, \qquad ...(5.6)$$

which implies that

$$(q_1 - q_2)d = 2im\pi \qquad ...(5.7)$$

where $m$ is an integer.

Equation (5.4) gives

$$q_{1,\,2} = \Bigg[ -\frac{\beta}{2}\Bigg\{(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\frac{\varepsilon v_0}{k_1} + M_0 n(1+\lambda n)$$

$$+ \frac{4\Omega^2 (1+\lambda n)^2 (zn+1)^2}{(1+\lambda n)(zn+1)n + (1+\lambda n)M_0 n + (1+\lambda_0 n)(zn+1)\frac{\varepsilon v_0}{k_1}}\Bigg\}$$

$$\pm \frac{1}{2}\Bigg[\beta^2\Bigg\{(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\frac{\varepsilon v_0}{k_1} + M_0 n(1+\lambda n)$$

$$+ \frac{4\Omega^2 (1+\lambda n)^2 (zn+1)^2}{(1+\lambda n)(zn+1)n + (1+\lambda n)M_0 n + (1+\lambda_0 n)(zn+1)\frac{\varepsilon v_0}{k_1}}\Bigg\}^2$$

$$+ 4k^2\Bigg\{(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\frac{\varepsilon v_0}{k_1} + M_0 n(1+\lambda n)$$

$$+ \frac{4\Omega^2 (1+\lambda n)^2 (zn+1)^2}{(1+\lambda n)(zn+1)n + (1+\lambda n)M_0 n + (1+\lambda_0 n)(zn+1)\frac{\varepsilon v_0}{k_1}}\Bigg\}$$

$$\Bigg\{(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\frac{\varepsilon v_0}{k_1} + M_0 n\,(1+\lambda n)$$

$$- \frac{g\beta(1+\lambda n)(zn+1)}{n}\Bigg\}\Bigg]^{1/2}$$

$$\left[(1+\lambda n)(zn+1)n+(1+\lambda_0 n)(zn+1)\frac{\varepsilon v_0}{k_1}+M_0 n(1+\lambda n)+\frac{4\Omega^2(1+\lambda n)^2(zn+1)^2}{(1+\lambda n)(zn+1)n+(1+\lambda n)M_0 n+(1+\lambda_0 n)(zn+1)\frac{\varepsilon v_0}{k_1}}\right]^{-1} \quad \ldots(5.8)$$

Inserting the values of $q_1$, $q_2$ from (5.8) in (5.7) and simplifying, we obtain

$$A_7 n^7+A_6 n^6+A_5 n^5+\ldots+A_2 n^2+A_1 n^1+A_0=0, \qquad \ldots(5.9)$$

where

$$A_7=(\beta^2 d^2+4m^2\pi^2+4k^2 d^2)\lambda^2 z^2,$$

$$A_6=(\beta^2 d^2+4m^2\pi^2+4k^2 d^2)\{2\lambda z(\lambda+z)+2\lambda\lambda_0 z^2\frac{\varepsilon v_0}{k_1}+2M_0\lambda^2 z\},$$

$$A_0=4k^2 d^2\,(-g\beta\,\varepsilon v_0/k_1), \qquad \ldots(5.10)$$

and the coefficients $A_1-A_5$ being quite lengthy and not needed in the discussion of stability have not been written here.

If $\beta<0$ (stable stratification), Eq. (5.9) does not admit of any positive root of n and the system is always stable for disturbances of all wave-numbers.

If $\beta>0$ (unstable stratification), the constant term in eq. (5.9) is negative. Equation (5.9), therefore, has atleast one positive real root and so the system is unstable for all wave numbers.

## REFERENCES

[1] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* Dover Publications, New York 1981.

[2] P.K. Bhatia and J.M. Steiner, *Z. Anew Math. Mech.* **52,** (1972) 321.

[3] P.K. Bhatia and J.M. Steiner, *J. Math. Anal. Appl.,* **41** (1973) 271.

[4] L.A. Eltayeb, *Z. Angew Math. Mech.,* **55** (1975), 599.

[5] B.A. Toms and D.J. Strawbridge, *Trans. Faraday Soc.* **49** (1953) 1225.

[6] J.W. Scanlon and L.A. Segel, *Physics of Fluids,* **16**, (1973), 573.

[7] R.C. Sharma and K.N. Sharma, *J. Math. Phys. Sci.,* **16** (1982) 167.

[8] J.G. Oldroyd, *Proc. Roy. Soc. (London),* A **245,** (1958), 278.

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# CONTOUR INTEGRALS ASSOCIATED WITH TWO GENERAL CLASS OF POLYNOMIALS FOX'S *H*-FUNCTION AND THE MULTI-VARIABLE *H*-FUNCTION

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## ABSTRACT

Here, we present two contour integrals based on a lemma, for the multivariable *H*-function [8] involving the product of Fox's *H*-function [1] and two general class of polynomials [6] with essentially arbitrary coefficients. By assigning suitable suitable values to the coefficients, the main result of two theorems can be reduced to integrals involving for instance the classical polynomials of Jacobi, Laguerre and Hermite. The multivariable *H*-function occuring in each of our main results can be reduced to the generalised Lauricella's hypergeometric functions of several complex variables.

**1. INTRODUCTION AND NOTATIONS** Srivastava and Panda [8] have introduced the multivariable *H*-function

$$H^{0,\,\lambda\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A,\,C\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}\left(\begin{matrix}[(a):\theta';\ldots;\theta^{(r)}]: \\ [(c):\psi';\ldots;\psi^{(r)}]:\end{matrix}\ \begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\ldots,z_r\right) \quad \ldots(1.1)$$

The defining integral of the above function, its various special cases and other details can be found in the paper referred to above. For the sake of brevity

$$\Lambda_i = \sum_{j=1}^{\lambda}\theta_j^{(i)} + \sum_{j=1}^{B^{(i)}}\phi_j^{(i)} - \sum_{j=1}^{C}\psi_j^{(i)} - \sum_{j=1}^{D^{(i)}}\delta_j^{(i)} < 0 \quad \ldots(1.2)$$

$$\Omega_i = -\sum_{j=\lambda+1}^{A}\theta_j^{(i)} + \sum_{j=1}^{v^{(i)}} - \sum_{j=1+v^{(i)}}^{B^{(i)}} - \sum_{j=1}^{C}\psi_j^{(i)}$$

$$+ \sum_{j=1}^{u^{(i)}}\delta_j^{(i)} - \sum_{j=1+u^{(i)}}^{D^{(i)}}\delta_j^{(i)} > 0, \ \forall\, i\ \varepsilon\ 1,\ldots,r. \quad \ldots(1.3)$$

**2. LEMMA** If $h > 0$, $c > -Re(y)$, $Re(\rho) > 0$, $m$, $m'$ are arbitrary positive integers and $A_{n,\alpha}$, $A_{n',\alpha'}$ are arbitrary constants, real or complex then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}\,(x+y)^{-\rho}\,S_n^m\,(l(x+y)^{-1})\,S_{n'}^{m'}\,(l'(x+y)^{-1})$$

$$.\,H_{P,Q}^{M,N}\left[\,y'(x+y)^{-\sigma}\,\middle|\,\begin{matrix}(e_p, E_p)\\(f_Q, F_Q)\end{matrix}\right]dt$$

$$= \sum_{g=1}^{M}\ \sum_{G=0}^{\infty}\ \sum_{\alpha}^{[n/m]}\ \sum_{\alpha'}^{[n'/m']}\ \frac{(-1)^G\,\phi(\eta_G)}{G!\,F_g}\,(y')\,\frac{\eta_G\,(-n)_{m\alpha}\,A_{n,\alpha}}{\alpha!}\,l^{\alpha}$$

$$\frac{(-n')_{m'\alpha'}}{\alpha'!}\,A_{n',\alpha'}\,(l')^{\alpha'}\,e^{-hy}\,\frac{h^{\rho+\alpha+\alpha'+\sigma\eta_G-1}}{\Gamma(\rho+\alpha+\alpha'+\sigma\eta_G)}, \qquad \ldots(2.1)$$

given that

$$\sigma > 0, E_j > 0, j = 1, \ldots, \rho;\ F_j > 0, j = 1, \ldots, Q;$$

and $|\arg(y')| < \dfrac{T\pi}{2}$, $T > 0$

$$\left(T = \sum_{j=1}^{n} E_j - \sum_{j=n+1}^{P} E_j + \sum_{j=1}^{m} F_j - \sum_{j=m+1}^{Q} F_j\right).$$

**Proof of the Lemma** The assertion (1) of the Lemma follows at once by applying the definition of Fox's $H$-function and multivariable $H$-function in conjunction with the well-known Hankel's contour integral for the Gamma function which is given in the following form

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}\,(x+y)^{-\sigma}\,dt = \frac{h^{\rho-1}\,e^{-hy}}{\Gamma(\rho)} \qquad \ldots(2.2)$$

$h > 0$, where for convergence $c > -Re(y)$ and $Re(\rho) > 0$.

**3. THE MAIN CONTOUR INTEGRAL** The main results of this paper are the contour integrals in the following theorem :

**Theorem-1** With $\Lambda_i$ and $\Omega_i$ defined by (1.2) and (1.3) respectively, let $\Lambda_i \le 0$ and $|\arg(z_i)| < \Omega_i\,\pi/2$, $\forall\ i = 1, \ldots, r$ ...(3.1)

where the both equalities hold for suitably restricted values of the complex variables $z_1, \ldots, z_r$. Now, the function $H_{P,Q}^{M,N}$ be defined by [4] and [5], general class of polynomials be defined by [6] and le $h > 0$ $c > -Re(y)$, then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}\,(x+y)^{-\rho}\, S_n^m\left(l(x+y)^{-1}\right) S_{n'}^{m'}\left(l'(x+y)^{-1}\right)$$

$$H_{P,Q}^{M,N}\left[\, y'\,(x+y)^{-\sigma} \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right] . H(z_1(x+y)^{-\rho_1}, \ldots, z_r(x+y)^{-\rho_r})\, dt$$

$$= \sum_{g=1}^{m} \sum_{G=0}^{\infty} \sum_{\alpha}^{[n/m]} \sum_{\alpha'}^{[n'/m']} \frac{(-1)^G\,\phi(\eta_G)}{G!\,F_g} (y')^{\eta_G} \frac{(-n)_{m\alpha} A_{n,\alpha}}{\alpha!} (l)^{\alpha}$$

$$\frac{(-n')_{m'\alpha'}}{\alpha'!} A_{n',\alpha'} (l')^{\alpha'}\, h^{\rho+\alpha+\alpha'+\sigma\eta_G^{-1}}\, e^{-hy}$$

$$H^{\,0,\lambda : (u',v') ;\ldots; (u^{(r)}, v^{(r)})}_{\,A, C+1 : [B', D'] ;\ldots; [B^{(r)}, D^{(r)}]} \left( \begin{matrix} [(a) : \theta' ;\ldots; \theta^{(r)}] : \\ [(c) : \psi' ;\ldots; \psi^{(r)}] , \end{matrix} \right.$$

$$[(b') : \phi'] ;\ldots; [(b^{(r)}) : \phi^{(r)}];$$

$$[1-\rho-\alpha-\alpha'-{}^{\sigma}\eta_G; \rho_1, \ldots, \rho_r] : [(d') : \delta'] ;\ldots; [(d^{(r)}) : \delta^{(r)}] ;$$

$$\left. z_1 h^{\rho_1}, \ldots, z_r h^{\rho_r} \right) \quad \ldots(3.2)$$

provided that $\rho_j > 0,\ \ j = 1, \ldots, r,\ \ \sigma > 0,\ \ |\arg y'| < \frac{T\pi}{2}$ and

$Re\,(\rho) > -c \sum_{j=1}^{r} \rho_j.$

**Theorem-2** Under the hypothesis preceding the assertion (3) of the theorem-1

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}\,(x+y)^{-\rho}\, H_{P,Q}^{M,N}\left[\, y'\,(x+y)^{-\sigma} \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$S_n^m\left(l(x+y)^{-1}\right) S_{n'}^{m'}\left(l'(x+y)^{-1}\right) H(z_1(x+y)^{\rho_1}, \ldots, z_r(x+y)^{\rho_r})\, dt$$

$$= \sum_{g=1}^{m} \sum_{G=0}^{\infty} \sum_{\alpha}^{[n/m]} \sum_{\alpha'}^{[n'/m']} \frac{(-1)^G\,\phi(\eta_G)}{G!\,F_g} (y')^{\eta_G} \frac{(-n)_{m\alpha} A_{n,\alpha}}{\alpha!} (l)^{\alpha}$$

$$\frac{(-n')_{m'\alpha'}}{\alpha'!} A_{n',\alpha'} (l')^{\alpha'}\, h^{\rho+\alpha+\alpha'+\sigma\eta_G-1}\, e^{-hy}$$

$$H^{\,0,\lambda : (u',v') ;\ldots; (u^{(r)}, v^{(r)})}_{\,A, C+1 : [B', D'] ;\ldots; [B^{(r)}, D^{(r)}]} \left( \begin{matrix} [(a) : \theta' ;\ldots; \theta^{(r)}] : \\ [(c) : \psi' ;\ldots; \psi^{(r)}] : \end{matrix} \right.$$

$$[\rho+\alpha+\alpha'+\sigma\eta_G; \rho_1, \ldots, \rho_r] : [(b') : \phi'] ;\ldots; [(b^{(r)}) : \phi^{(r)}] ;$$

$$[(d') : \delta'] ;\ldots; [(d^{(r)}) : \delta^{(r)}] ;$$

$$\left. z_1 h^{-\rho_1}, \ldots, z_r h^{-\rho_r} \right) \quad \ldots(3.3)$$

provided that $\rho_j > 0, \quad j = 1, \ldots, r, \quad \sigma > 0, \quad |\arg y'| < \dfrac{T\pi}{2}$ and $Re\,(\rho) > -c \sum_{j=1}^{r} \rho_j$.

**Proofs of Theorem 1 and 2.** The contour integrals formula of Theorem 1 can be proved by replacing the multivariable $H$-function in the integrand by its multiple contour integral [7]. Now we change the order of integration, evaluate the innermost integral by appealing to the assertion (2) of the lemma and then interpret the resulting multiple contour integral as an $H$-function of $r$-complex variables.

**4. APPLICATIONS** The contour integral formulas of Theorem 1 and 2 of the preceding section possesses manifold generality. The multivariable $H$-function may be transformed into $G$-functions, $E$-functions, Lauricella's functions, Hypergeometric functions, Legendre functions, Bessel functions, Hermite functions, Laguerre, functions and other functions in one, two or more arguments.

On other hand we can get several integral formulas for such special polynomials $\left( S_n^m\,[y] \right)_{n=0}^{\infty}$ by giving the values to $n$ and $n'$.

**Particular cases of Lemma** ($I$) By applying our Lemma to the case of Hermite polynomials ([10], p.106, Eqn. (5.5.4) and [9]) by setting $S_n^2(x) \rightarrow x^{n/2} H_n\,(\frac{1}{2\sqrt{x}})$ in which case $m = 2$, $A_{n,s} = (-1)^s$ and also letting $m' = 2$, $A_{n',s'} = (-1)^{s'}$, we obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}\,(x+y)^{-\rho}\,.\,H_{P,Q}^{M,N}\left[\, y'(x+y)^{-\sigma} \,\middle|\, \begin{matrix}(e_p, E_p)\\(f_Q, F_Q)\end{matrix}\right]$$

$$\left[(l(x+y)^{-1})^{n/2} H_n\!\left(\frac{1}{\sqrt{l(x+y)^{-1}}}\right)\right]\left[(l'(x+y)^{-1})^{n'/2}\, H_{n'}\!\left(\frac{1}{\sqrt{l'\,(x+y)^{-1}}}\right)\right] dt$$

$$= \sum_{g=1}^{m} \sum_{G=0}^{\infty} \sum_{s=0}^{[n/2]} \sum_{s'=0}^{[n'/2]} \frac{(-1)^G\,\phi(\eta_G)}{G!\,F_g}\,(y')^{\eta_G}\,\frac{(-n)_{2s}\,2^{n}}{s!}\left(\frac{-l}{4}\right)^{s}$$

$$\frac{(-n')_{2s'}\,2^{n'}}{s'!}\left(\frac{-l'}{4}\right)^{s'} \frac{e^{-hy}\,h^{\rho+\sigma\eta_G+s+s'-1}}{\Gamma(\rho+{}^{\sigma}\eta_G+s+s')} \qquad \ldots(4.1)$$

which holds under the same conditions as those required for (1).

(*II*) For the Laguerre polynomials ([10], p.101, Eqn.(5.1.6) and [9]) setting $S_n(x) \to L_n^{\alpha}(x)$ in which case $m = 1$ and $A_{n,s}\binom{n+\alpha}{n}\left(\frac{1}{(\alpha+1)_k}\right)$ and also taking $m' = 1$ and $A_{n',s'} = \binom{n'+\alpha'}{n'}\left(\frac{1}{(\alpha'+1)_{k'}}\right)$ we obtain the Lemma in the following form

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}(x+y)^{-\rho} \,.\, H_{P,Q}^{M,N}\left[ y'(x+y)^{-\sigma} \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$\left[\sum_{s=0}^{n} \binom{n+\alpha}{n-s} \frac{(-l(x+y)^{-1})^s}{s!}\right]\left[\sum_{s'=0}^{n'} \binom{n'+\alpha'}{n'-s'} \frac{(-l'(x+y)^{-1})^s}{s'!}\right] dt$$

$$= \sum_{g=1}^{m} \sum_{G=0}^{\infty} \sum_{s=0}^{n} \sum_{s'=0}^{n'} \frac{(-1)^s \phi(\eta_G)}{G!\, F_g} (y')^{\eta_G} \frac{e^{-hy} h^{\rho+\sigma\eta_G+s+s'-1}}{\Gamma(\rho+\sigma\eta_G+s+s')}$$

$$\binom{n+\alpha}{n-s}\frac{(-x)^s}{s!}\binom{n'+\alpha'}{n'-s'}\frac{(-x')^{s'}}{s'!} \qquad \text{...(4.2)}$$

which holds under the same conditions as those required for (1).

(*III*) For Lauricella's functions taking $\lambda = A$, $u^{(i)} = 1$, $v^{(i)} = B^{(i)}$, $D^{(i)} =$ $\forall\, i = 1, \ldots, r$, we obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}(x+y)^{-\rho} \,.\, H_{P,Q}^{M,N}\left[ y'(x+y)^{-\sigma} \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_Q, f_Q) \end{matrix} \right]$$

$$S_n^m\left(l(x+y)^{-1}\right) S_{n'}^{m'}\left(l'(x+y)^{-1}\right)$$

$$F^{A\,:\,B';\ldots;B^{(r)}}_{C\,:\,D';\ldots;D^{(r)}}\left(\begin{matrix}[1-(a):\theta';\ldots;\theta^{(r)}]: \\ [1-(c):\psi';\ldots;\psi^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[1-(b'):\phi'];\ldots;[1-(b^{(r)}):\phi^{(r)}];\, z_1(x+y)^{-\rho_1} \\ \vdots \\ [1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}];\, z_2(x+y)^{-\rho_r}\end{matrix}\right) dt$$

$$= \sum_{g=1}^{m} \sum_{G=0}^{\infty} \sum_{\alpha=0}^{[n/m]} \sum_{\alpha'=0}^{[n'/m']} \frac{(-1)^G \phi(\eta_G)}{G!\, F_g} (y')^{\eta_G} h^{\sigma\eta_G+\alpha+\alpha'+\rho-1} e^{-hy}$$

$$\frac{(-n)_{m\alpha}}{\alpha!} A_{n,\alpha}(l)^{\alpha} \frac{(-n')_{m'\alpha'} A_{n',\alpha'}}{\alpha'!} (l')^{\alpha'}$$

$$. F \begin{matrix} A : B' ;...; B^{(r)} \\ C+1 : D' ;...; D^{(r)} \end{matrix} \left( \begin{matrix} [1-(a) : \theta' ;...; \theta^{(r)}] : \\ [1-(c) : \psi' ;...; \psi^{(r)}], \end{matrix} \right.$$

$$[1-(b') : \phi'] ;...; [1-(b^{(r)}) : \phi^{(r)}];$$

$$[1-\rho+\sigma\eta_G-\alpha-\alpha' : \rho_1 ,... \rho_r] : [1-(d') : \delta'] ;...; [1-(d^{(r)}) : \delta^{(r)}];$$

$$\left. z_1 h^{\rho_1} ;...; z_r h^{\rho_r} \right) \quad ...(4.3)$$

valid under the same conditions as given in the equation (4).

(*IV*) For Hermite polynomial in the theorem (1), we get

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{hx} (x+y)^{-\rho} . H_{P,Q}^{M,N} \left[ y'(x+y)^{-\sigma} \middle| \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$\left[ (l(x+y)^{-1})^{n/2} H_n \left( \frac{1}{\sqrt{l(x+y)^{-1}}} \right) \right] \left[ (l'(x+y)^{-1})^{n'/2} H_{n'} \left( \frac{1}{\sqrt{l'(x+y)^{-1}}} \right) \right] dt$$

$$. H (z_1 (x+y)^{-\rho_1} ,... z_r (x+y)^{-\rho_r}] \, dt$$

$$= \sum_{g=1}^{M} \sum_{G=0}^{\infty} \sum_{s=0}^{[n/2]} \sum_{s'=0}^{[n'/2']} \frac{(-1)_G \, \phi(\eta_G)}{G! \, F_g} (y')^{\eta_G} h^{\sigma\eta_G + s + s' + \rho - 1} e^{-hy}$$

$$\frac{(-n)_{2s} \, 2^n}{s!} \left( \frac{-l}{4} \right)^s \frac{(-n')_{2s'} \, 2^{n'}}{s'!} \left( \frac{-l'}{4} \right)^{s'}$$

$$H \begin{matrix} 0, \lambda : (u', v') ;...; (u^{(r)}, v^{(r)}) \\ A, C+1 : [B', D'] ;...; [B^{(r)}, D^{(r)}] \end{matrix} \left( \begin{matrix} [(a) : \theta' ;...; \theta^{(r)}] : \\ [(c) : \psi' :...; \psi^{(r)}] : \end{matrix} \right.$$

$$[(b') : \phi'] ;...; [(b^{(r)}) : \phi^{(r)}] ;$$

$$[1-\rho-\sigma\eta_G - s - s' ; \rho_1 ,..., \rho_r] : [(d') : \delta'] ;...; [(-d^{(r)}) : \delta^{(r)}] ;$$

$$\left. \begin{matrix} z_1 h^{\rho_1} \\ : \\ z_2 h^{\rho_2} \end{matrix} \right) \quad ...(4.4)$$

which holds under the same conditions as given in equation (4).

(*V*) Also by applying our theorem 1 to the case of Laguerre polynomials, we obtain

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{hx} (x+y)^{-\rho} . H_{P,Q}^{M,N} \left[ y'(x+y)^{-\sigma} \middle| \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$L_n^{(\alpha)}(l(x+y)^{-1})\,L_{n'}^{(\alpha')}(l'(x+y)^{-1}).H(z_1(x+y)^{-\rho_1},\ldots,z_r(x+y)^{-\rho_r}]dt$$

$$= \sum_{g=1}^{m}\sum_{G=0}^{\infty}\sum_{s=0}^{[n]}\sum_{s'=0}^{[n']}\frac{(-1)^G\,\phi(\eta_G)}{G!\,F_g}(y')^{\eta_G}\,h^{\sigma\eta_G+s+s'-1}\,e^{-hy}\,dt$$

$$\frac{(-l)^s}{s!}\binom{n+\alpha}{n-s}\frac{(-l')^{s'}}{s'!}\binom{n'+\alpha'}{n'-s'}$$

$$H^{\,0,\,\lambda\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{\,A,\,C+1\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}\left(\begin{matrix}[(a):\theta';\ldots;\theta^{(r)}]:\\ [(c):\psi';\ldots;\psi^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix} & [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [1-\rho+\sigma\eta_G-s-s';\rho_1,\ldots,\rho_1]: & [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\begin{matrix} z_1\,h^{\rho_1}\\ \vdots\\ z_2\,h^{\rho_r}\end{matrix}\right) \quad \ldots(4.5)$$

valid for the same conditions as given in equation (4).

(*VI*) Letting $n' \to 0$, the above five results of particular cases reduce to the known results obtained by Chaurasia and Tyagi [3].

(*VII*) Letting $n' \to 0$, $n \to 0$ the above five results of particular cases reduce to the known results obtained by V.B.L. Chaurasia [2].

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## REFERENCES

[1] B.L.J. Braaksma, Asymptotic expansions and analytic continuations for a class of Barnes-integrals, *Compositio Math.*, **15** (1961), 339-341.

[2] V.B.L. Chaurasia, Contour integrals involving Fox's H-function and the multivariable H-functions, *Istanbul Univ. Fen Fak. Mecm. Ser. A* **47** (1983-1986), 111-115.

[3] V.B.L. Chaurasia and S.Tyagi, Contour integrals involving a general class of polynomials Fox's H-function and the multivariable H-function, *Bull. Math. Soc. Sci. Math. R.S. Romanie (N.S.)* (to appear).

[4] C. Fox, The G and H functions as symmetrical fourier kernels, *Trans. Amer. Math. Soc.*, **98** (1961), 395-429.

[5] P. Skibinski, Some expansion theorem for the H-function, *Ann. Polon. Math.*, **23** (1970), 125-138.

[6] H.M. Srivastava, A contour integral involving Fox's H-function, *Indian J. Math.*, **14** (1972), 1-6.

[7] H.M. Srivastava and R. Panda, Expansion theorems for the H-function of several complex variables, *J. Reine Angew. Math.,* **288** (1976), 129-145.

[8] H.M. Srivastava and R. Panda, Some bilateral generating functions for a class of generalised hypergeometric polynomials, *J. Reine Angew. Math.,* **283/284** (1976), 265-274.

[9] H.M. Srivastava and N.P. Singh, The integration of certain products of the multivariable H-function with a general class of polynomials, *Rend. Circ. Mat. Plaermo* (2), **32** (1983) 157-187.

[10] G. Szegö, Orthogonal Polynomials, *Amer. Math. Soc. Colloq. Publ.* **23** fourth edition, Amer. Math. Soc. Providence, Rhode Island, 1975.

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# USE OF THE MULTIVARIABLE *H*-FUNCTION AND GENERAL POLYNOMIALS IN LINEAR FLOW OF HEAT IN AN ANISOTROPIC FINITE SOLID MOVING IN A CONDUCTING MEDIUM

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## ABSTRACT

The use of special functions and their applications is being increasing important in technical applications. Space research, nuclear reactor, wood technology, soil mechanics and the mechanics of solids of fibrous structure give rise to several problems of the application of special functions.

The aim of this paper is to discuss the application of certain product of Fox's H-function, general polynomials, the generalized prolate spheroidal wave function and the multivariable *H*-function in obtaining formal solution of the partial differential equation related to a problem of liner flow of heat in an anisotropic finite solid moving in a conducting medium. Thus, the present study unifies and extends a number of results scattered in the literature.

**1. INTRODUCTION** We shall consider a problem of linear heat flow in an anisotropic finite solid moving in a conducting medium, given by following partial differential equation (see [6])

$$\frac{\partial}{\partial y}\left[ (1-y^2)\frac{\partial U}{\partial y}\right] - \frac{pcv}{\lambda'}\frac{\partial U}{\partial y} + \frac{Q(y)}{\lambda'} = \frac{pc}{\lambda'}\frac{\partial U}{\partial t} \qquad ...(1.1)$$

with the law of conductivity $K = \lambda'(1-y^2)$, $Q(y)$ is the intensity of a continuous source of heat situated inside this solid.

Let the initial temperature of the rod be given by

$$U(y, 0) = F(y) \qquad ...(1.2)$$

assuming

$v = (\rho - \sigma)/q,$

$$Q(y) = -(\rho + \sigma)\,\lambda' y\,\frac{\partial U}{\partial y} - (s', y)^2\,\lambda' U,$$

$$q = pc/\lambda'$$

a solution of (1.1) takes the form

$$U(y, t) = \sum_{l=0}^{\infty} B_l \exp(-T_l t)\,\phi_l^{\rho,\sigma}(s', y) \qquad \text{...(1.3)}$$

where $B_l = (l + p)(l + p + \rho + \sigma + 1)/q$.

Consider

$$F(y) = (1-y)^{\alpha-\rho}(1+y)^{\beta-\sigma} H_{P,Q}^{M,N}\left[ z(1-y)^h (1+y)^k \,\middle|\, \begin{matrix} (e_P, E_P) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$S_{n_1,\dots,n_s}^{m_1,\dots,m_s}\left[ (1-y)^{\alpha_1 h_1}(1+y)^{\alpha_1 k_1}, \dots, (1-y)^{\alpha_s h_s}(1+y)^{\alpha_s k_s} \right]$$

$$H_{A,\,C\,:\,[B',\,D'];\dots;[B^{(r)},\,D^{(r)}]}^{0,\,\lambda\,:\,(u',\,v');\dots;(u^{(r)},\,v^{(r)})}$$

$$\left( z_1(1-y)^{h'}(1+y)^{k'}, \dots, z_r(1-y)^{h^{(r)}}(1+y)^{k^{(r)}} \right) \qquad \text{...(1.4)}$$

where the multivariable $H$-function of several complex variables introduced by Srivastava and Panda ([8] and [9]; see also [10], p.251). The results needed in our investigations are as follows :

The general polynomials $S_{n_1,\dots,n_s}^{m_1,\dots,m_s}[x_1,\dots,x_s]$ occuring in the integral stands for the multivariable polynomials given by Srivastava ([11], p.185, eqn. (7)] defined and represented in the following slightly modified from

$$S_{n_1,\dots,n_s}^{m_1,\dots,m_s}[x_1,\dots,x_s] = \sum_{\alpha_1=0}^{[n_1/m_1]} \dots \sum_{\alpha_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1\alpha_1}}{\alpha_1!} \dots \frac{(-n_s)_{m_s\alpha_s}}{\alpha_s!}$$

$$A[n_1, \alpha_1;\dots;\dots;n_s, \alpha_s]\, x_1^{\alpha_1} \dots, x_s^{\alpha_s} \qquad \text{...(1.5)}$$

where $n_i = 0, 1, 2, \dots$; $m_i \neq 0$ $(i = 1, \dots, s)$, $m_i$ is an arbitrary positive integer. The coefficients $A[n_i, \alpha_i;\dots;n_s, \alpha_s]$ being arbitrary constants, real or complex.

If we take $s = 1$ in equation (1.5) and denote $A[n, \alpha]$ thus obtained by $A_{n,\alpha'}$, we arrive at the well-known general class of polynomials $S_n^m[x]$ introduced by Srivastava ([12], p.1, eqn.(1)).

The series representation of Fox's $H$-function ([3] and [7]; see also [10], p.12) :

$$H_{P,Q}^{M,N}\left[ z \;\middle|\; \begin{matrix}(e_p, E_p) \\ (f_Q, F_Q)\end{matrix} \right] = \sum_{g=1}^{M} \sum_{G=0}^{\infty} \frac{(-1)^G \phi(\eta_G) z^{\eta_G}}{G!\, F_g} \qquad \text{...(1.6)}$$

where

$$\phi(\eta_G) = \prod_{j=1, j \neq g}^{M} \Gamma(f_j - F_j \eta_G) \prod_{j=1}^{N} \Gamma(1 - e_j - E_j \eta_G)$$

$$\left\{ \prod_{j=M+1}^{Q} \Gamma(1 - f_j - F_j \eta_G) \prod_{j=N+1}^{P} \Gamma(e_j - E_j \eta_G) \right\}^{-1}$$

and $\eta_G = (f_g + G)/F_g$ ...(1.7)

The solution of the following differential equation

$$(1 - y^2) \frac{d^2 x}{dy^2} + [(\sigma - \rho) - (\rho + \sigma + 2)y] \frac{dx}{dy} + [(s') + (s'y)^2]x = 0 \qquad \text{...(1.8)}$$

is the generalized prolate spheroidal wave function ([4], p.107, Eqn. (2.11)), which is denoted as

$$\phi_n^{\rho,\sigma}(s', y) = \sum_{a=0}^{\infty} R_{q,n}^{\rho,\sigma}(s')\, P_{a+n}^{(\rho,\sigma)}(y) \qquad \text{...(1.9)}$$

where $s' = 0$, $\xi(0) = (n + a)(\rho + \sigma + n + a + 1)$, $a \geq 0$.

The orthogonality property of the generalized prolate spheroidal wave function ([4], p.107, eqn.(3.1))

$$\int_{-1}^{1} (1 - y)^{\rho} (1 + y)^{\sigma} \phi_v^{\rho,\sigma}(s', y)\, \phi_w^{\rho,\sigma}(s'y)\, dx = N_{v,w}^{\rho,\sigma} \delta_{v,w} \qquad \text{...(1.10)}$$

where

$$N_{v,w}^{\rho,\sigma} = \sum_{a=0}^{\infty} R_{a,v}^{\rho,\sigma}(s')^2$$

$$\frac{2^{\rho+\sigma+1}\Gamma(v + \rho + a + 1)\Gamma(v + \sigma + a + 1)}{(2v + 2a + \rho + \sigma + 1)\Gamma(v + a + 1)\Gamma(v + a + \rho + \sigma + 1)}$$

and $\delta_{v,w}$ is the Kronecker delta.

## 2. THE MAIN INTEGRAL

$$\int_{-1}^{1} (1 - y)^{\alpha}(1 + y)^{\beta} \phi_w^{\rho,\sigma}(s', y)\, H_{P,Q}^{M,N}\left[ z(1 - y)^h (1 + y)^k \;\middle|\; \begin{matrix}(e_p, E_p) \\ (f_Q, F_Q)\end{matrix} \right]$$

$$S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}[(1-y)^{\alpha_1 h_1}(1+y)^{\alpha_1 k_1},\ldots,(1-y)^{\alpha_s h_s}(1+y)^{a_s k_s}]$$

$$H_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}$$

$$\left(z_1(1-y)^{h'}(1+y)^{k'},\ldots,z_r(1-y)^{h^{(r)}}(1-y)^{h^{(r)}}(1+y)^{k^{(r)}}\right)dy$$

$$=\sum_{g=1}^{M}\sum_{v,G=0}^{\infty}\sum_{\alpha_1=0}^{[n_1/m_1]}\cdots\sum_{\alpha_s=0}^{[n_s/m_s]}\frac{(-1)^G\phi(\eta_G)}{G!\,F_g}z^{\eta_G}R_{v,w}^{\rho,\sigma}(s',y)$$

$$\frac{(-n_1)_{m_1\alpha_1}}{\alpha_1!}\cdots\frac{(-n_s)_{m_s\alpha_s}}{\alpha_s!}A(n_1,\alpha_1;\ldots;n_s,\alpha_s)$$

$$2^{1+\alpha+\beta+(h+k)\eta_G+(h_1+k_1)\alpha_1,\ldots,(h_s+k_s)\alpha_s}$$

$$\sum_{u=0}^{v+w}(-v+w)_u(\rho+\sigma+v+w+1)_u\{(\rho+1)_u\,u!\}^{-1}$$

$$H_{A+2,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,\lambda+2:(u',v');\ldots;(u^{(r)},v^{(r)})}$$

$$\left(\begin{array}{l}[(a):\theta',\ldots,\vartheta^{(r)}],[-u-\alpha-h\eta_G-h_1\alpha_1,\ldots,h_s\alpha_s;h',\ldots,h^{(r)}\},\\ [(c):\psi',\ldots,\psi^{(r)}];\end{array}\right.$$

$$[-\beta-k\eta_G-k_1\alpha_1,\ldots,k_s\alpha_s:k',\ldots,k^{(r)}]:$$
$$[-1-\alpha-\beta-u-(h+k)\eta_G-(h_1+k_1)\alpha_1\ldots-(h_s+k_s)\alpha_s:$$

$$\left.\begin{array}{l}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\\ h'+k',\ldots,h^{(r)}+k^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{array}z_1 2^{h'+k'},\ldots,z_r 2^{h^{(r)}+k^{(r)}}\right)\qquad\ldots(2.1)$$

where $\eta_i = 0, 1, 2, \ldots; m_i \neq 0$ $[i = 1, \ldots s]$ $m_i$ is an arbitrary positive integer. The cofficients $A[n_1, \alpha_1;\ldots;n_s, \alpha_s]$ being arbitrary constant real or complex.

$$Re(\alpha + hf_l/F_l + \sum_{i=1}^{r} h^{(i)}\, d_j^{(i)}/\delta_j^{(i)}) > -1$$

$$Re(\beta + kf_l/F_l + \sum_{i=1}^{r} k^{(i)}\, d_j^{(i)}/\delta_j^{(i)}) > -1,\ Re(\rho) > -1,\ Re(\sigma) > -1,$$

$$h^{(i)} > 0,\ k^{(i)} > 0,\ l = 1, \ldots, M;\ i = 1, \ldots, r, j = 1, \ldots, u^{(i)},\ T_i > 0$$

$$|\arg(z_i)| < T_i\,\pi/2$$

$$|\arg(z)| < T\pi/2 \text{ and } T = \left(\sum_{1}^{N} E_i - \sum_{N+1}^{P} E_i + \sum_{1}^{M} F_i - \sum_{M+1}^{Q} F_i\right).$$

Evaluation of (2.1). The integral (2.1) can be evaluated in a routine manner with the help of result Chaurasia ([10], p.31 eqn. (2.1) and (1.5)).

## 3. SOLUTION OF THE PROBLEM

$$U(y, t) = \sum_{g=1}^{M} \sum_{v, G=0}^{\infty} \sum_{\alpha_1=0}^{[n_1/m_1]} \cdots \sum_{\alpha_s=0}^{[n_s/m_s]} \frac{(-1)^G \phi(\eta_G)}{G!\,F_g}\, z^{\eta_G} R_{v,\,l}^{\rho,\,\sigma}(s', y)$$

$$2^{1+\alpha+\beta+(h+k)\eta_G+(h_1+k_1)\alpha_1, \ldots, (h_s+k_s)\alpha_s}$$

$$\sum_{l,\,u=0}^{v} \frac{(-v)_u(\rho+\sigma+v+1)_u}{(\rho+1)_u\, u} \{N_{l,\,1}^{\rho,\,\sigma}\}^{-1} \phi_{\lambda}^{\rho,\,\sigma}(s', y)$$

$$e^{-v(v+\rho+\sigma+1)t/q}\, H_{A+2,\, C+1 : [B', D'] ;\ldots; [B^{(r)}, D^{(r)}]}^{0,\, \lambda+2 : (u', v') ;\ldots; (u^{(r)}, v^{(r)})}$$

$$\left(\begin{matrix} [-u-\alpha-h\eta_G-h_1\alpha_1, \ldots, -h_s\alpha_s; h', \ldots, h^{(r)}], [-\beta-k\eta_G-k_1\alpha_1, \ldots, k_s\alpha_s; k', \ldots k^{(r)}], \\ [(c) : \psi', \ldots, \psi^{(r)}], [-1-\alpha-\beta-u-(h+k)\eta_G-(h_1+k_1)\alpha_1 \ldots -(h_s+k_s)\alpha_s : \end{matrix}\right.$$

$$\left.\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'] ;\ldots; [b^{(r)} : \phi^{(r)}] ; \\ : h'+k', \ldots, h^{(r)}+k^{(r)}] : [(d') : \delta'] ;\ldots; [(d^{(r)}) : \delta^{(r)}]; \end{matrix}\ z_1 2^{h'+k'}, \ldots, z_r 2^{h^{(r)}+k^{(r)}}\right) \quad \ldots(3.1)$$

valid under the same conditions as given in (2.1).

**Evaluation of (3.1).** Take $t = 0$ then by (1.3), we have

$$F(y) = \sum_{l=0}^{\infty} B_l \phi_l^{\rho,\,\sigma}(s', y), \quad \ldots(3.2)$$

where $F(y)$ is given by (1.4).

Equation (3.2) is valid since $F(y)$ is continuous and bounded variation in the interval (-1, 1).

Multiplying both sides of (3.2) by $(1-x)^{\rho}(1+x)^{\sigma} \phi_{w}^{\rho,\,\sigma}(s', y)$ $\rho > -1, \sigma > -1$ and integrating between -1 to +1 with respect to y and using the orthogonality property (1.9) a lemma ([5], p.57, eqn. (2)) and integral (2.1) after simplification, we arrive at the required result.

## 4. SPECIAL CASES

(*i*) If we take $s = 1, m_1 = 2, A_{n_1, \alpha_1} = (-1)^{\alpha_1}$ in (2.1), we obtain

$$\int_{-1}^{1} (1-y)^{\alpha}(1+y)^{\beta}\, \phi_w^{\rho,\sigma}(s', y)\, H_{P,Q}^{M,N}\left[ z(1-y)^h(1+y)^k \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$(1-y)^{h_1 n_1/2}\,(1+y)^{k_1 n_1/2}\, H_{n_1}\left[ \frac{1}{2\sqrt{(1-y)^{h_1}(1+y)^{k_1}}} \right]$$

$$H^{\,0, \lambda : (u', v') ;\ldots; (u^{(r)}, v^{(r)})}_{\,A, C : [B', D'] ;\ldots; [B^{(r)}, D^{(r)}]}$$

$$\left( z_1(1-y)^{h'}\,(1+y)^{k'} ,\ldots, z_r\,(1-y)^{h^{(r)}}\,(1-y)^{h^{(r)}}\,(1+y)^{k^{(r)}} \right) dy$$

$$\sum_{g=1}^{M}\ \sum_{v, G=0}^{\infty}\ \sum_{\alpha_1=0}^{[n_1/2]} \frac{(-1)^G \phi\,(\eta_G)}{G!\, F_g}\, z^{\eta_G}\, R_{v,w}^{\rho,\sigma}(s', y)\, \frac{(-n_1)_{2\alpha_1}\,(-1)^{\alpha_1}}{\alpha_1!}$$

$$2^{1+\alpha+\beta+(h+k)\eta_G+(h_1+k_1)\alpha_1} \sum_{u=0}^{v+w} (-v+w)_u(\rho+\sigma+v+w+1)_u$$

$$\{(\rho+1)_u\, u!\}^{-1}\, H^{\,0, \lambda+2 : (u', v') ;\ldots; (u^{(r)}, v^{(r)})}_{\,A+2, C+1 : [B', D'] ;\ldots; [B^{(r)}, D^{(r)}]}$$

$$\left( \begin{matrix} [-u-\alpha+h\eta_G-h_1\alpha_1 : h' ,\ldots, h^{(r)}],\ [-\beta-k\eta_G-k_1\alpha_1 ; k', \ldots, k^{(r)}], \\ [(c) : \psi' ,\ldots, \psi^{(r)}].\ [-1-\alpha-\beta-u-(h+k)\eta_G-(h+k)-(h_1+k_1)\alpha_1 : h'+k' ,\ldots, h^{(r)}+k^{(r)}] : \\ [(a) : \theta' ,\ldots, \theta^{(r)}] : [(b') : \phi'] ;\ldots; [(b^{(r)}) : \phi^{(r)}]; \\ : [(d') : \delta'] ;\ldots; [(d^{(r)}) : \delta^{(r)}]; \end{matrix} \quad z_1\, 2^{h'+k'} ,\ldots, z_r\, 2^{h^{(r)}+k^{(r)}} \right) \quad \ldots(4.1)$$

valid under the same conditions as given by (2.1).

(*ii*) On taking $s = 1, m_1 = 1$ and $A_{n_1, \alpha_1} \binom{f_1+f'}{n_1} \dfrac{(f'+g'+n_1+1)_{\alpha_1}}{(f'+1)_{\alpha_1}}$

in (2.1), we get

$$\int_{-1}^{1} (1-y)^{\alpha}(1+y)^{\beta}\, \phi_w^{\rho,\sigma}(s', y)\, H_{P,Q}^{M,N}\left[ z(1-y)^h(1+y)^k \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$P_{n_1}^{(f', g')}\,[1-2(1-y)^{h_1}\,(1+y)^{k_1}]\, H^{\,0, \lambda : (u', v') ;\ldots; (u^{(r)}, v^{(r)})}_{\,A, C : [B', D'] ;\ldots; [B^{(r)}, D^{(r)}]}$$

$$\left( z_1(1-y)^{h'}(1+y)^{k'}, \ldots, z_r(1-y)^{h^{(r)}}(1-y)^{h^{(r)}}(1+y)^{k^{(r)}} \right) dy$$

$$\sum_{g=1}^{M} \sum_{v,\,G=0}^{\infty} \sum_{\alpha_1=0}^{n_1} \frac{(-1)^G \phi(\eta_G)}{G!\,F_g} z^{\eta_G} R_{v,\,w}^{\rho,\,\sigma}(s', y) \frac{(-n_1)_{\alpha_1}}{\alpha_1!} \binom{n_1+f'}{n_1}$$

$$\frac{(f'+g'+n_1+1)_{\alpha_1}}{(f'+1)_{\alpha_1}} \,.\, 2^{1+\alpha+\beta+(h+k)\eta_G+(h_1+k_1)\alpha_1}$$

$$\sum_{u=0}^{v+w} (-v+w)_u (\rho+\sigma+v+w+1)_u \{(\rho+1)_u\, u!\}^{-1}$$

$$H^{\,0,\,\lambda+2\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{\,A+2,\,C+1\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}$$

$$\left( \begin{matrix} [-u-\alpha-h\eta_G-h_1\alpha_1 \,;\, h', \ldots, h^{(k)}],\ [-\beta-k\eta_G-k_1\alpha_1 : k', \ldots, k^{(r)}] \\ [(c):\psi', \ldots, \psi^{(r)}],\ [-1-\alpha-\beta-u-(h+k)\eta_G-(h_1+k_1)\alpha_1 : h'+k', \ldots, h^{(r)}+k^{(r)}]: \end{matrix} \right.$$

$$\left. \begin{matrix} [(a):\theta', \ldots, \theta^{(r)}] : [(b'):\phi'] \,;\ldots;\, [b^{(r)}:\phi^{(\rho)}] \,; \\ [(d'):\delta'] \,;\ldots;\, [(d^{(r)}):\delta^{(r)}]; \end{matrix} \quad z_1\, 2^{h'+k'}, \ldots, z_r\, 2^{h^{(r)}+k^{(r)}} \right) \qquad \ldots(4.2)$$

valid under the same conditions as required by (2.1).

(*iii*) Let $s = 1$, $m_1 = 1$ and $A_{n_1,\,\alpha_1} \binom{n_1+f'}{n_1} \frac{1}{(f'+1)_{\alpha_1}}$ in (2.1), we get

$$\int_{-1}^{1} (1-y)^{\alpha}(1+y)^{\beta}\, \phi_w^{\rho,\,\sigma}(s', y)\, H_{P,Q}^{M,N}\left[ z(1-y)^h(1+y)^k \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$L_{n_1}^{(\alpha_1)} (1-y)^{h_1}(1+y)^{k_1}\, H^{\,0,\,\lambda\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{\,A,\,C\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}$$

$$\left( z_1(1-y)^{h'}(1+y)^{k'}, \ldots, z_r(1-y)^{h^{(r)}}(1-y)^{h^{(r)}}(1+y)^{k^{(r)}} \right) dy$$

$$\sum_{g=1}^{M} \sum_{G=0}^{\infty} \sum_{\alpha_1=0}^{n_1} \frac{(-1)^G \phi(\eta_G)}{G!\,F_g} z^{\eta_G} R_{v,\,w}^{\rho,\,\sigma}(s', y) \frac{(-n_1)_{\alpha_1} \binom{n_1+f'}{n_1}}{\alpha_1!\,(f'+1)_{\alpha_1}}$$

$$2^{1+\alpha+\beta+(h+k)\eta_G+(h_1+k_1)\alpha_1} \sum_{u=0}^{v+w} (-v+w)_u (\rho+\sigma+v+w+1)_u$$

$$\{(\rho+1)_u\, u!\}^{-1}\, H^{0,\,\lambda+2\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A+2,\,C+1\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}$$

$$\left(\begin{matrix}[-u-\alpha-h\eta_G-h_1\alpha_1 : h',\ldots,h^{(r)}],\ [-\beta-k\eta_G-k_1\alpha_1 : k',\ldots,k^{(r)}],\ [(a):\theta',\ldots,\theta^{(r)}]: \\ [(c):\psi',\ldots,\psi^{(r)}],\ [-1-\alpha-\beta-u-(h+k)\eta_G-(h_1+k_1)\alpha_1 : h'+k',\ldots,h^{(r)}+k^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1\, 2^{h'+k'},\ldots,z_r\, 2^{h^{(r)}+k^{(r)}}\right) \qquad \ldots(4.3)$$

valid under the same conditions as required by (2.1).

(*iv*) On taking $\lambda = A$, $u^{(i)} = 1$, $v^{(i)} = B^{(i)}$ and $D^{(i} = D^{(i)} + 1$, ∀ $i = 1, \ldots, r$ in (2.1), we obtain

$$\int_{-1}^{1}(1-y)^{\alpha}(1+y)^{\beta}\phi_w^{\rho,\,\sigma}(s',y)\, H_{P,Q}^{M,N}\left[ z(1-y)^h(1+y)^k \,\middle|\, \begin{matrix}(e_p, E_p) \\ (f_Q, F_Q)\end{matrix}\right]$$

$$S^{m_1,\ldots,m_s}_{n_1,\ldots,n_s}[(1-y)^{\alpha_1 h_1}(1+y)^{\alpha_1 k_1},\ldots,(1-y)^{\alpha_s h_s}(1+y)^{a_s k_s}]$$

$$F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left(\begin{matrix}[1-(a):\theta',\ldots,\theta^{(r)}]:[1-(b'):\phi'];\ldots; \\ [1-(c):\psi',\ldots,\psi^{(r)}]:[1-(d');\ldots;\end{matrix}\right.$$

$$\left.\begin{matrix}[1-(b^{(r)}):\phi^{(r)}] \\ [1-(d^{(r)}):\delta^{(r)}]\end{matrix}\ z_1(1-y)^{h'}(1+y)^{k'},\ldots,z_r(1-y)^{h^{(r)}}(1+y)^{k^{(r)}}\right)dx$$

$$\sum_{g=1}^{M}\ \sum_{v,\,G=0}^{\infty}\ \sum_{\alpha_1=0}^{[n_1/m_1]}\cdots\sum_{\alpha_s=0}^{[n_s/m_s]}\frac{(-1)^G\phi(\eta_G)}{G!\,F_g}\, z^{\eta_G}\, R^{\rho,\,\sigma}_{v,\,w}(s',y)$$

$$\frac{(-n_1)_{m_1\alpha_1}}{\alpha_1!}\cdots\frac{(-n_s)_{m_s\alpha_s}}{\alpha_s!}A(n_1,\alpha_1;\ldots;n_s,\alpha_s)$$

$$2^{1+\alpha+\beta+(h+k)\eta_G+(h_1+k_1)\alpha_1,\ldots,(h_s+k_s)\alpha_s}\sum_{u=0}^{v+w}(-v+w)_u$$

$$(\rho+\sigma+v+w+1)_u\,\{(\rho+1)_u\,u!\}^{-1}\,F^{A+1:B',\ldots,B^{(r)}}_{C:D',\ldots,D^{(r)}}$$

$$\left(\begin{matrix}[1+u+\alpha+h\eta_G+h_1\alpha_1+\ldots+h_s\alpha_s; h',\ldots,h^{(r)}],[1+\beta+k\eta_G+k_1\alpha_1+\ldots+k_s\alpha_s; k',\ldots,k^{(r)}] \\ [1-(c):\psi',\ldots,\psi^{(r)}]:\end{matrix}\right.$$

$$\begin{array}{l} [1-(a):\theta',\ldots,\theta^{(r)}]: \\ [2+\alpha+\beta+u+(h+k)\eta_G+(h_1+k_1)\alpha_1,\ldots,(h_s+k_s)\alpha_s : h'+k',\ldots,h^{(r)}+k^{(r)}]: \\ \left.\begin{array}{l}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}];\end{array} z_1 2^{h'+k'},\ldots,z_r 2^{h'+k'},\ldots,z_r 2^{h^{(r)}+k^{(r)}}\right) \end{array} \quad \ldots(4.4)$$

valid under the same conditions as given in (2.1).

(*v*) If we take $s = 1$ and $m_1 = 0$ in (2.1) it reduces to a known result of Chaurasia ([1], p.31, eqn.(2.1)).

(*vi*) If $m_1 = m_2 = \ldots = m_s = 0$ and $M = 1$, $N = p$, $P = p$, $Q = q + 1$ and $h \to 0$, $k \to 0$ and after a little simplification the result in (2.1) reduces to a known result recently obtained by Chaurasia and Gupta ([2], p.78, eqn. (8)).

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## REFERENCES

[1] V.B.L. Chaurasia, Linear flow of heat in an Anisotropic finite solid moving in a conducting medium and multivariable H-function, *Jñānābha* **20** (1990), 29-33.

[2] V.B.L. Chaurasia and V.G. Gupta, Some results on the *H*-function of several complex variables involving the generalized hypergeometric function and the generalized prolate spheroidal wave function, *Jñānābha* **13** (1983), 73-85.

[3] C. Fox, The *G* and *H* functions as symmetrical Fourier Kernels, *Trans. Amer. Math. Soc.* **98** (1961), 395-429.

[4] R.K. Gupta, Generalized prolate spheroidal wave function, *Proc. Indian Acad. Sci. Sect. A* **85** (1977), 104-114.

[5] E.D. Rainville, *Special Functions,* Macmillan, New York, 1960.

[6] V.P. Saxena and A.R. Nageria, Linear flow of heat in an anisotropic finite solid moving in a conducting medium *Jñānābha Sect. A* **4** (1974), 1-6.

[7] F. Skilinski, Some expansion theorems for the H-function, *Ann. Polon. Math.* **23** (1970), 123-138.

[8] H.M. Srivastava and R. Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials, *J. Reine Angew. Math.* **283/284** (1976), 265-274.

[9] H.M. Srivastava and R. Panda, Expansion theorems for the H- function of several complex variables, *J. Reine Angew. Math.* **288** (1976), 129-145.

[10] H.M. Srivastava, K.C. Gupta and S.P. Goyal, *The H-Functions of One and Two Variables with Applications,* South Asian Publishers, New Delhi and Madras, 1982.

[11] H.M. Srivastava, A multilinear generating function for the Konhauser sets of biorthogonal polynomials suggested by the Laguerre polynomials, *Pacific J. Math.* **117** (1985), 183-191.

[12] H.M. Srivastava, A contour integral involving Fox's H-function, *Indian J. Math.* **14** (1972), 1-6.

[13] H.M. Srivastava and N.P. Singh, The integration of certain products of the multivariable H-function with a general class of polynomials, *Rend. Circ. Mat. Palermo. Ser. II* **32** (1983), 157- 187.

[14] H.M. Srivastava and M.C. Daoust, On Eulerian integrals associated with Kampe' de Fériet's function, *Publ. Inst. Math. (Beograd) (N.S.)* **9**(23) (1969), 199-202.

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# CERTAIN INTEGRALS INVOLVING HYPERGEOMETRIC FUNCTIONS OF THREE AND FOUR VARIABLES

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## ABSTRACT

In the present paper some integrals involving hypergeometric functions of three and four variables are evaluated. Some particular cases are of interest.

## 1. INTRODUCTION

Let

$$E_\alpha f(\alpha) = f(\alpha+1),\ E_\alpha^m f(\alpha) = f(\alpha+m) \qquad \text{...(1.1)}$$

Recently Agarwal [1] employed the operator $E_\alpha$ to evaluate the integrals

$$\int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\,\Gamma(\beta-x)}\, {}_1F_1\left[\begin{matrix}\alpha'; \\ & v \\ \alpha+x;\end{matrix}\right] {}_1F_1\left[\begin{matrix}\beta'; \\ & v \\ \beta-x;\end{matrix}\right] dx;$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\, {}_1F_1\left[\begin{matrix}\alpha'+\beta'; \\ & 2v \\ \alpha+\beta-1;\end{matrix}\right] \qquad \text{...(1.2)}$$

$\mathrm{Re}(\alpha+\beta) > 1$, $n$ is an integer.

$$\int_{-\infty}^{\infty} \frac{1}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)}\, {}_1F_1\left[\begin{matrix}\alpha+\beta-1; \\ & v \\ \alpha+x;\end{matrix}\right] {}_1F_1\left[\begin{matrix}\gamma+\delta-1; \\ & v \\ \gamma+x;\end{matrix}\right] dx$$

$$= \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)}$$

$$\cdot {}_2F_2\left[\begin{matrix}\dfrac{\alpha+\beta+\gamma+\delta-3}{2}, & \alpha+\beta+\gamma+\delta-3; \\ \beta+\gamma-1, & \delta+\alpha-1;\end{matrix}\; 4v\right] \qquad \ldots(1.3)$$

$\mathrm{Re}(\alpha+\beta+\gamma+\delta) > 3.$

Ragab and Simary [7] have also evaluated these integrals, by a different technique.

In the present paper, following integrals are evaluated, which involve hypergeometric functions of three and four variables :

$$(I) \quad \int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\Gamma(\beta-x)}$$

$$H_B(\beta+\gamma, \gamma+\alpha, \alpha+\beta; \gamma, \alpha+x, \beta-x; u, v, w)\,dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} H_A(\beta+\gamma, \gamma+\alpha, \alpha+\beta; \gamma, \alpha+\beta-1; u, 2v, 2w)$$

$$\mathrm{Re}(\alpha+\beta) > 1,$$

$$(II) \quad \int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\Gamma(\beta-x)}$$

$$F_K(\gamma', \alpha+\beta, \alpha+\beta, \gamma+\beta, \alpha', \gamma+\beta; \gamma, \alpha+x, \beta-x; u, v, w)dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_M(\gamma', \alpha+\beta, \alpha+\beta, \gamma+\beta, \alpha', \gamma+\beta, \gamma, \alpha+\beta-1; u, 2v, 2w)$$

$$\mathrm{Re}(\alpha+\beta) > 1.$$

$$(III) \quad \int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\Gamma(\beta-x)}$$

$$F_A^{(3)}(\alpha', \gamma, \alpha+\beta, \gamma', \alpha+x, \beta-x; u, v, w)\,dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_G(\alpha', \alpha', \alpha', \gamma, \alpha, \beta; \gamma', \alpha+\beta-1, \alpha+\beta-1; u, 2v, 2w)$$

$$\mathrm{Re}(\alpha+\beta) > 1.$$

$$(IV) \quad \int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\Gamma(\alpha+x)\Gamma(\beta-x)}$$

$$F_{29}^{(4)}\left(\begin{matrix}\gamma+\alpha, \gamma+\alpha, \beta+\delta, \beta+\delta, \alpha+\delta, \gamma+\beta, \alpha+\delta, \gamma+\beta; \\ \gamma+\delta', \alpha+x, \beta-x, \gamma+\delta';\end{matrix}\; u, v, w, t\right)dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}$$

$$F_{58}^{(4)}\left(\begin{matrix}\gamma+\alpha, \gamma+\alpha, \beta+\delta, \beta+\delta, \alpha+\delta, \gamma+\beta, \alpha+\delta, \gamma+\beta; \\ \gamma+\delta', \alpha+\beta-1, \alpha+\beta-1, \gamma+\delta';\end{matrix}\; u, 2v, 2w, t\right)$$

$$\mathrm{Re}(\alpha+\beta) > 1.$$

$$(V)\quad \int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\Gamma(\beta-x)}$$

$$F_{28}^{(4)}\left(\begin{matrix}\gamma+\delta, \gamma+\delta, \alpha+\beta, \alpha+\beta, \gamma+\alpha, \delta+\beta, \gamma+\alpha, \delta+\beta; \\ \alpha+x, \beta-x, \gamma+\delta', \gamma+\delta';\end{matrix}\; u, v, w, t\right)dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}$$

$$.F_{47}^{(4)}\left(\begin{matrix}\gamma+\delta, \gamma+\delta, \alpha+\beta, \alpha+\beta, \gamma+\alpha, \delta+\beta, \gamma+\alpha, \delta+\beta; \\ \gamma+\delta', \gamma+\delta', \alpha+\beta-1, \alpha+\beta-1;\end{matrix}\; u, v, 2w, 2t\right)$$

$$\operatorname{Re}(\alpha+\beta) > 1.$$

$$(VI)\quad \int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\Gamma(\alpha+x)\Gamma(\beta-x)}$$

$$.F_{27}^{(4)}\left(\begin{matrix}\gamma+\alpha, \gamma+\alpha, \delta+\beta, \delta+\beta, \gamma'+\alpha', \gamma'+\alpha', \delta'+\beta', \delta'+\beta'; \\ \gamma+\delta, \alpha+x, \gamma+\delta, \beta-x;\end{matrix}\; u, v, w, t\right)dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}$$

$$.F_{47}^{(4)}\left(\begin{matrix}\gamma+\alpha, \gamma+\alpha, \beta+\delta, \beta+\delta, \gamma+\alpha', \gamma'+\alpha', \delta'+\beta', \delta'+\beta'; \\ \gamma+\delta, \alpha+\beta-1, \gamma+\delta, \alpha+\beta-1;\end{matrix}\; u, 2v, w, 2t\right)$$

$$\operatorname{Re}(\alpha+\beta) > 1.$$

where $H_A, H_B, H_C$ are Srivastava's functions of three variables [cf. [9]; p.43], $F_G, F_K, F_M, F_A^{(3)}$ are Lauricella's functions of three variables [cf. [9]; p. 42, 33] and $F_{27}^{(4)}, F_{28}^{(4)}, F_{29}^{(4)}, F_{27}^{(4)}, F_{57}^{(4)}, F_{58}^{(4)}$ are Lauricella's functions of four variables [cf. [8]; p. 125-129].

In what follows, we will employ the following results [cf. [2]; p.224].

$$\int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \qquad ...(1.4)$$

## 2. ANALYSIS OF THE RESULTS

To prove ($I$), multiply both sides of (1.4) by

$\dfrac{\Gamma(\gamma+\alpha)\Gamma(\alpha+\beta)\Gamma(\beta+\gamma)}{\Gamma(\gamma)}$ and apply the operator

$\exp(uE_{\alpha}E_{\alpha'} + vE_{\beta}E_{\beta'} + wE_{\gamma}E_{\gamma'})$; we obtain

$$\int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]\;\Gamma(\gamma+\alpha)\Gamma(\beta+\gamma)}{\sin(\pi x)\;\Gamma(\alpha+x)\;\Gamma(\beta-x)\Gamma(\gamma)}$$

$$\sum_{m,n,p=0}^{\infty} \frac{(\beta+\gamma)_{m+p}(\gamma+\alpha)_{m+n}(\alpha+\beta)_{n+p}}{(\gamma)_m\,(\alpha+x)_m\,(\beta-x)_n} \frac{u^m}{m!}\frac{v^n}{n!}\frac{w^p}{p!}\,dx$$

$$= \frac{2^{\alpha+\beta-2}\,\Gamma(\gamma+\alpha)\Gamma(\alpha+\beta)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta-1)\Gamma(\gamma)}$$

$$\sum_{m,n,p=0}^{\infty} \frac{(\beta+\gamma)_{m+p}(\gamma+\alpha)_{m+n}(\alpha+\beta)_{n+p}}{(\gamma)_m\,(\alpha+\beta-1)_{m+n}} \frac{u^m}{m!}\frac{(2v)^n}{n!}\frac{(2w)^p}{p!}\,dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} H_A\,(\beta+\gamma,\gamma+\alpha,\alpha+\beta;\gamma,\alpha+\beta-1;u,2v,2w)$$

which completes the proof of (*I*).

Now consider (1.4), multiply both the sides by

$\dfrac{\Gamma(\gamma+\alpha)\Gamma(\beta+\delta)\Gamma(\gamma+\beta)\Gamma(\alpha+\delta)}{\Gamma(\gamma'+\delta')}$ and operate both the sides by

$\exp\,(uE_\gamma E_{\gamma}' + vE_\alpha E_{\alpha'} + wE_\beta E_{\beta'} + tE_\delta E_{\delta'})$, we get

$$\int_{-\infty}^{\infty} \frac{\sin\,[(2n+1)\pi x]\,\Gamma(\gamma+\alpha)\Gamma(\beta+\delta)\Gamma(\gamma+\beta)\Gamma(\alpha+\delta)}{\sin\,(\pi x)\,\Gamma(\alpha+x)\,\Gamma(\beta-x)\Gamma(\gamma'-\delta')}$$

$$\sum_{m,n,p,q=0}^{\infty} \frac{(\gamma+\alpha)_{m+n}(\beta+\delta)_{p+q}(\gamma+\beta)_{m+p}(\alpha+\delta)_{n+q}}{(\gamma'+\delta')_{m+q}\,(\alpha+x)_n\,(\beta-x)_p} \frac{u^m}{m!}\frac{v^n}{n!}\frac{w^p}{p!}\frac{t^q}{q!}dx$$

$$= \frac{2^{\alpha+\beta-2}\,\Gamma(\gamma+\alpha)\Gamma(\beta+\delta)\Gamma(\gamma+\beta)\Gamma(\alpha+\delta)}{\Gamma(\alpha+\beta-1)\Gamma(\gamma'+\delta')}$$

$$\sum_{m,n,p,q=0}^{\infty} \frac{(\gamma+\alpha)_{m+n}(\beta+\delta)_{p+q}(\gamma+\beta)_{m+p}(\alpha+\delta)_{n+q}}{(\gamma'+\delta')_{m+q}\,(\alpha+\beta-1)_{n+p}} \frac{u^m}{m!}\frac{(2v)^n}{n!}\frac{(2w)^p}{p!}\frac{t^q}{q!}$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}$$

$$F_{58}^{(4)}\left(\begin{matrix}\gamma+\alpha,\gamma+\alpha,\beta+\delta,\beta+\delta,\gamma+\beta,\gamma+\beta,\alpha+\delta,\alpha+\delta;\\ \gamma'+\delta',\alpha+\beta-1,\alpha+\beta-1,\gamma'+\delta';\end{matrix}\;u,v,2w,t\right)$$

which is the required result (*IV*).

By employing the operators

$$\exp\,(uE_\gamma E_{\gamma}' + vE_\alpha E_{\alpha'} + wE_\beta E_{\beta'}),$$
$$\exp\,(uE_\gamma E_{\gamma}' + vE_\alpha + wE_\beta E_{\beta'})E_{\alpha'}],$$
$$\exp\,(uE_\gamma E_{\gamma}' + vE_\delta E_{\delta'} + wE_\alpha E_{\alpha'} + tE_\beta E_{\beta'})$$

and $\exp\,(uE_\gamma E_{\gamma}' + vE_\alpha E_{\alpha'} + wE_\delta E_{\delta'} + E_\beta E_{\beta'})$ to (1.4).

we can easily obtain results (*II*), (*III*), (*V*) and (*VI*).

## 3. PARTICULAR CASES

By setting $u = 0$ in $(I)$, following interested results are obtained

$$\int_{-\infty}^{\infty} \frac{[\sin (2n+1)\pi x]}{\sin (\pi x)\, \Gamma(\alpha+x)\Gamma(\beta-x)}$$

$$.\, F_2 (\alpha+\beta, \alpha+\beta, \gamma+\alpha, \beta+\gamma; \alpha+x, \beta-x; v, w)\, dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_1(\alpha+\beta, \alpha+\beta, \gamma+\alpha, \beta+\gamma; \alpha+\beta-1, \alpha+\beta-1; 2v, 2w) \quad ...(3.1)$$

For $w = 0$; $(II)$ reduces to the following form

$$\int_{-\infty}^{\infty} \frac{[\sin (2n+1)\pi x]}{\sin (\pi x)\, \Gamma(\alpha+x)\Gamma(\beta-x)}$$

$$.\, F_3 (\gamma', \alpha+\beta, \gamma+\beta, \alpha'; \gamma, \alpha+x; u, v)\, dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_3(\gamma', \alpha+\beta, \gamma+\beta, \alpha'; \gamma, \alpha+\beta-1; u, 2v) \quad ...(3.2)$$

For $u = 0$; $(II)$ reduces to the following interesting case

$$\int_{-\infty}^{\infty} \frac{[\sin (2n+1)\pi x]}{\sin (\pi x)\, \Gamma(\alpha+x)\Gamma(\beta-x)}$$

$$.\, F_2(\alpha+\beta, \alpha+\beta, \alpha', \gamma+\beta, \alpha+x; \beta-x; v, w)\, dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_2(\alpha+\beta, \alpha+\beta, \alpha', \gamma+\beta; \alpha+\beta-1, \alpha+\beta-1; 2v, 2w) \quad ...(3.3)$$

Finally, if we set $u = 0$ in $(III)$, we obtain

$$\int_{-\infty}^{\infty} \frac{[\sin (2n+1)\pi x]}{\sin (\pi x)\, \Gamma(\alpha+x)\Gamma(\beta-x)}$$

$$.\, F_2 (\alpha', \alpha', \alpha, \beta; \alpha+x, \beta-x; v, w)\, dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_1(\alpha', \alpha', \alpha, \beta; \alpha+\beta-1, \alpha+\beta-1; 2v, 2w) \quad ...(3.4)$$

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## REFERENCES

[1] B.M. Agrawal, Application of Δ and E operators to evaluate certain integrals, *Proc. Camb. Phil. Soc.* **64**, (1968), 99-104.

[2] Bateman Project, *Higher Transcendental Functions,* Vol. I, 1953.

[3] Bateman Project. *Tables of Integral Transforms,* Vol. II, 1954.

[4] S.S. Bhati, Certain Integrals Involving Hypergeometric Functions of two variables, *Jñānābha Sect. A.* **5**, (1975), 75-80.

[5] B.R. Bhonsle, On some results involving Jacobi polynomila, *J. Indian Math. Soc.* **26** (1962), 187-190.

[6] J.L. Burchnall and T.W. Chaundy, Expansions of Appell's double hypergeometric functions-II, *Quart. J. Math. Oxford Ser.* **12** (1941), 112-128.

[7] F.M. Ragab and M.A. Simary, Integrals involving the products of generalized Whittaker functions, *Proc. Camb. Phil. Soc.* **61** (1965), 429-432.

[8] C. Sharma and C.L. Parihar, Hypergeometric functions of four variables (I), *J. Indian Acad. Math.* **11**, No. 2 (1989), 121-133.

[9] H.M. Srivastava and Per W. Karlsson, *Multiple Gaussian Hypergeometric Series,* Ellis Horwood Limited, Chichester, New York, (1985).

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# LAPLACE TRANSFORM OF GENERALIZED HYPERGEOMETRIC DISTRIBUTION FUNCTION ASSOCIATED WITH ZONAL POLYNOMIAL

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## ABSTRACT

In this paper authors first discuss probability density functions of generalized hypergeometric function of symmetric positive definite matrix associated with zonal polynomial, then establish new result, which has a wide range of application in mathematical Statistics by virtue of Laplace transform of matrix variate.

**1. INTRODUCTION.** At first Herz (1955) has introduced the theory of matrix argument involving certain special functions such as Bessel function, confluent hypergeometric function etc. Later on James (1961) extended the theory of special functions with matrix argument involving zonal polynomial. James (1961) and Constantine (1963) have given series of papers on multivariate analysis. Mathai (1978) introduced Laplace transform of binomial distribution function for positive definite matrix. Recently Sethi and Mishra (1995) discussed 'Laplace Transform of the binomial distribution function ${}_1F_0\,(\alpha; - ;X)$ associated with zonal polynomial' of symmetric matrix. In this paper generalized hypergeometric distribution function associated with zonal polynomial has been used to define new formula by virtue of Laplace transform of matrix argument. It is very interesting that result earlier by the authors Sethi and Mishra (1995) and Mathai (1991) follows a special case.

All the matrices are assumed to $p \times p$ order symmetric positive definite. Unless otherwise stated. $A > 0$ means $A$ is positive definite.

$R(.)$ means that real part of (.). $\int_0^I (.)\, dA$ means that (.) is integrated out over the $p \times p$ symmetric matrix $A$, such that $A > 0$ and $I - A > 0$ that in all the eigenvalues of $A$ are between 0 and 1. The notation $\|(.)\|$ stands for the norms of (.). Here since the matrices are symmetric positive definite the largest eigenvalues can be taken as the norms. $|(.)|$ stands for determination of (.), $tr(.)$ denotes the *trace* of (.).

**2. ZONAL POLYNOMIALS.** Let $S$ be a positive definite symmetric $p \times p$ matrix and $\phi(S)$ a polynomial in the element of $S$. Then the transformation

$$\phi(S) \rightarrow \phi(L^{-1} SL'^{-1}) \quad L \in GL(P) \qquad \text{... (2.1)}$$

defines a representation of the real linear group $GL(p)$ in the vector space of all polynomials in $S$. The space $V_k$ of homogeneous polynomial of degree $K$, in invariant under the given transformation and decomposes into the direct sum of irreducible subspace

$$V_k = \Sigma_k + V_{k,\,k}$$

where $K = (k_1, k_2, ..., k_p); k_1 \geq k_2 \geq ... \geq k_p \geq 0$ sums over all partitions of k into not more than $p$ parts.

These subspaces are generated by the zonal polynomial $Z(S)$, being invariant under the orthogonal group i.e.

$$Z_k\,(HSH') = Z_k(S)\ ,\ H \in 0(p)\ .$$

They are homogeneous symmetric polynomials in the characteristic roots of $S$.

The fundamental property of the zonal polynomials is given by the following integral,

$$\int_{o(p)} C_k\,(H'TSH)\,dH = C_k(S)\,C_k(T)/C_k(I) \qquad \text{... (2.2)}$$

where $I$ is the identity matrix of $dH$, is invariant Haar measure on the othogonal group.

Hypergeometric function of matrix will be defined as certain series of zonal polynomials. The equivalance of the two sets of functions follows from the important integral identity.

$$\int_{S>0} \exp\,(-\,tr\,RS)\ |S|^{-t-(p+1)/2}\ C_k\,(ST)\,dS$$

$$= \Gamma_p\,(t, k)\,C_k\,(R^{-1}\,T)\ |R|^{-t} \qquad \text{... (2.3)}$$

where $C_k(S)$ in the zonal polynomial corresponding to the partitions $k = (k_1, k_2, ..., k_p); k_1 \geq k_2 \geq ... \geq k_p \geq 0$ of the integer $k$, $t$ is the complex number satisfying $R(t) > \frac{(p-1)}{2}$ and $\Gamma_p(t, k) = \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(t + k_i - \frac{(i-1)}{2}\right)$, the integration is over the space of positive definite symmetric $p \times p$ matrices.

## 3. SOLUTION OF LAPLACE TRANSFORM OF THE DISTRIBUTION FUNCTION ${}_pF_q$

**Theorem .** If $X$, $T$ and $B$ are real positive definite symmetric matrix of order $p \times p$. If $k = (k_1, k_2, \ldots, k_p)$; $k_1 \geq k_2 \geq \ldots \geq k_p \geq 0$ runs over all partitions into not more than $p$ parts, $C_k(S)$ the corresponding zonal polynomial. The Laplace transform of the generalized hypergeometric function ${}_pF_q$ is

$$L_f(T) = \frac{|T+B|^{-t}\, {}_pF_q\, [\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; (T+B)^{-1}]}{|B|^{-t}\, {}_pF_q\, [\alpha_1, \ldots \alpha_p; \beta_1, \ldots, \beta_q; (B^{-1})]} \qquad \ldots (3.1)$$

here the distribution function $f(x)$ is given as

$$f(x) = \frac{|X|^{t-(p+1)/2}\, e^{-tr(BX)}\, {}_pF_q\, [\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; X]}{\Gamma_p(t, k)\, |B|^{-t}\, {}_pF_q\, [\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; B^{-1}]} \qquad \ldots (3.2)$$

**Proof :** We consider the integral

$$\int_{x>0} |X|^{t - \frac{(p+1)}{2}}\, e^{-tr(BX)}\, {}_pF_q\, [\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; X]\, dX \qquad \ldots (3.3)$$

where the generalized hypergeometric function ${}_pF_q$ in terms of zonal polynomial defined as

$$\begin{aligned} {}_pF_q\, &[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; X] \\ &= \sum_{k=0}^{\infty} \sum_{k} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k\, k!}\, C_K(X) \end{aligned} \qquad \ldots (3.4)$$

here the zonal polynomial $C_K(X)$ is the component of $e^{tr(X)^k}$.

Then the equation (3.3) becomes

$$\begin{aligned} &\int_{X>0} |X|^{t-(p+1)/2}\, e^{-tr(BX)}\, {}_pF_q\, [\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; X]\, dX \\ &= \int_{X>0} |X|^{t-(p+1)/2}\, e^{-tr(BX)} \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k\, k!}\, C_K(X)\, dX \end{aligned} \qquad \ldots (3.5)$$

$$= \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, k!} \int_{X>0} |X|^{t-(p+1)/2} e^{-tr(BX)} C_k(X) dX$$

By the virtue of the integral

$$\int_{X>0} |X|^{t-\frac{(p+1)}{2}} e^{-tr(BX)} C_K(X)\, dX = \Gamma_p(t, k)\, |B|^{-t} C_K(B^{-1})$$

the equation (3.5) becomes

$$= \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, k!} \Gamma_p(t, k)\, |B|^{-t} C_K(B^{-1})$$

$$= \Gamma_p(t, k)\, |B|^{-t} \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, K!} C_K(B^{-1})$$

$$= \Gamma_p(t, k)\, |B|^{-t} {}_pF_q[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; B^{-1}] \qquad \ldots (3.6)$$

Now *p.d.f.* is defined as

$$f(x) = \frac{|X|^{t-(p+1)/2} e^{-tr(BX)} {}_pF_q[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; X]}{\Gamma_p(t, k)\, |B|^{-t} {}_pF_q[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; B^{-1}]}$$

for $X = X' > 0$, $B = B' > 0$, $\| X \| < 1$, $\| B \| < 1$ and $R(t) > (p-1)/2$
$= 0$, elsewhere $\qquad \ldots (3.7)$

$$f(X) \geq 0 \;\forall X \quad \int f(X)\, dX = 1$$

taking Laplace transform on both sides of equation (3.7), we have

$$\int_{X>0} e^{-tr(TX)} f(X)\, dX = \int_{X>0} e^{-tr(TX)} \frac{|X|^{t-(p+1)/2} e^{-tr(BX)}}{\Gamma_p(t, k)\, |B|^{-t}} \times \frac{{}_pF_q[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q X]\, dX}{{}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; B^{-1})}$$

$$L_f(T) = \eta \int_{X>0} |X|^{t-(p+1)/2} e^{-tr(T+B)X} {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; X)\, dX$$

where $L_f(T) = \int_{x>0} e^{-tr(TX)} f(X)\, dX$

and $\eta = [\Gamma_p(t, k)\, |B|^{-t} {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; B^{-1}]^{-1}$

or $$L_f(T) = \eta \int_{X>0} |X|^{t-(p+1)/2} e^{-tr(T+B)X}$$

$$\sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, k!} C_K(X)\, dX$$

$$L_f(T) = \eta \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, k!} \int_{X>0} |X|^{t-(p+1)/2} \times e^{-tr\,(T+B)X} C_K(X)\, dX \quad \ldots (3.8)$$

Putting

$(T+B)X = Y \quad \Rightarrow \quad tr(T+B)X = tr(Y)$

$X = (T+B)^{-1} Y \quad \Rightarrow \quad dX = |T+B|^{-(p+1)/2}\, dY$

$|X| = |T+B|^{-1} |Y| \Rightarrow |X|^{t-(p+1)/2} = |T+B|^{-t+(p+1)/2} |Y|^{t-(p+1)/2}$

when $X = 0 \quad \Rightarrow \quad Y = 0$.

Putting these values in equation (3.8) it becomes

$$L_f(T) = \eta \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, k!} \int_{Y>0} |T+B|^{-t+(p+1)/2} \times |Y|^{t-(p+1)/2} e^{-tr\,(Y)} C_K[(T+B)^{-1} Y]\; |T+B|^{-(p+1)/2}\, dY$$

or
$$L_f(T) = \eta \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, k!} \int_{Y>0} |Y|^{-t+(p+1)/2} \times e^{-tr(Y)} C_K[(T+B)^{-1} Y]\, dY \quad \ldots (3.9)$$

Constantine [6] has proved a integral relating zonal polynomial with gamma function as

$$\int_{X>0} e^{-tr(RX)} |X|^{t-(p+1)/2} C_K(TX)\, dX = \Gamma_p(t,k)\, C_K(R^{-1} T)\, |R|^{-t}$$

This yields the result $R = I$ this equation gives

$$\int_{X>0} |X|^{t-(p+1)/2} e^{-tr(X)} C_K(XT)\, dX = \Gamma_p(t,k)\, C_K(T) \quad \ldots (3.10)$$

using the result (3.10) in equation (3.9), we get

$$L_f(T) = \eta \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, k!} \frac{|T+B|^{-t}}{k!} \Gamma_p(t,k)\, C_k[(T+B)^{-1}]$$

Putting the value of η we get

$$L_f(T) = \frac{|T+B|^{-t}\,\Gamma_p(t,k) \sum_{k=0}^{\infty} \sum_{K} \dfrac{(\alpha_1)_k \ldots (\alpha_p)_k\, C_K\,[(T+B)^{-1}]}{(\beta_1)_k \ldots (\beta_q)_k \quad k!}}{\Gamma_p\,(t,k)\ |B|^{-1}\ {}_pF_q\,[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; B^{-1}]}$$

$$L_f(T) = \frac{|T+B|^{-t}\ {}_pF_q\,[\alpha_1, \ldots, \alpha_p\,;\, \beta_1, \ldots, \beta_q; (T+B)^{-1}]}{|B|^{-t}\ {}_pF_q\,[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; B^{-1}]} \ldots (3.11)$$

or $$L_f(T) = \left|\frac{T+B}{B}\right|^{-t} \chi$$

where $\chi = \dfrac{{}_pF_q\,[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; (T+B)^{-1}]}{{}_pF_q\,[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; B^{-1}]}$

or $$L_f(T) = |I + B^{-1}\,T|^{-t}$$

$$= |I + B^{-1/2}\,T\,B^{-1/2}|^{-t}$$

The moment generating function of $f(X)$ is

$$|I + B^{-1}\,T|^{-t} \text{ and } I - B^{-1}\,T > 0.$$

## 4. SPECIAL CASES

(1) When we put $p = 0$, $q = 0$ in equation (3.11), the equation reduces to exponential series

$$L_f(T) = \frac{|T+B|^{-t}}{|B|^{-t}}\ \frac{{}_0F_0\,[(T+B)^{-1}]}{{}_0F_0\,[(B)^{-1}]}.$$

(2) When $p = 1$ and $q = 0$ equation (3.11) is reduced to the Laplace transform of binomial function ${}_1F_0\,(\alpha\,; -X)$ as

$$L_f(T) = \frac{|T+B|^{-t}}{|B|^{-t}}\ \frac{{}_1F_0\,[\alpha\,;-\,;(T+B)^{-1}]}{{}_1F_0\,[\alpha\,;-\,;(B^{-1})]}$$

(3) When $p = 1$ and $q = 1$, equation (3.11) is reduced to the Laplace transform of confluent hypergeometric function ${}_1F_1\,(\alpha; \gamma\,; X)$ as

$$L_f(T) = \frac{|T+B|^{-t}}{|B|^{-t}}\ \frac{{}_1F_1\,[\alpha; \gamma; (T+B)^{-1}]}{{}_1F_1\,[\alpha; \gamma; (B^{-1})]}.$$

(4) When $p = 2$ and $q = 1$ equation (3.11) reduces to the Laplace transform of Gaussian hypergeometric function ${}_2F_1\,(\alpha, \beta; \gamma; X)$ as

$$L_f(T) = \frac{|T+B|^{-t}\ {}_2F_1\,[\alpha, \beta; \gamma; (T+B)^{-1}]}{|B|^{-t}\ {}_2F_1\,[\alpha, \beta; \gamma; (B^{-1})]}$$

When $K = 0$ in all the investigations, we have probability density for gamma function due to Mathai and Saxena [4]

$$f(X) = \frac{|X|^{t-(p+1)/2}\, e^{-tr\,(BX)}}{\Gamma_p(t)\ |B|^{-t}}$$

where $X = X' > 0$ and $B = \beta' > 0$ and its Laplace transform is given as

$$L_f(T) = |I + B^{-1}\,T|^{-t} = |I + B^{-1/2}\,TB^{-1/2}|^{-t}$$

The moment generating function of $f(X)$ is

$$|I + B^{-1}T|^{-t}; I - B^{-1}\,T > 0.$$

## REFERENCES

[1] A.M. *Mathai, Indian J. Pure and Applied Math.* **22** (1991), 887.

[2] H. Exton, *Multiple Hypergeometric Functions and Applications,* Ellis Horwood, Chichester (1976).

[3] H.M. Srivastava and Per. W. Karlson, *Multiple Gaussian Hypergeometric series*, Ellis. Horwood, Chicherster (1985).

[4] A.M. Mathai and R.K. Saxena, *The H-Function with Applications in Statistics and other Disciplines,* Wiley Eastern limited, New Delhi India P. 96 (1978).

[5] A.M.Mathai : *Maths Nachr.* **84** (1978), 174.

[6] A.G.Constentine : *Some non-central distribution problem in multivariate analysis Ann. Math. Statis.* **34** (1963), 1270-85.

[7] A.T.James : *Zonal polynomial of the real positive definite symmetric matrices Ann. Math.* **74** (1961), 456-69.

[8] K. Subranmaniam on the zonal polynomial and other functions of matrix arguments value **38** series A. (1976) Pt. 33, 221-258.

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# EFFECT OF SUSPENDED PARTICLES ON THERMAL INSTABILITY OF A ROTATING MAXWELLIAN VISCOELATIC FLUID IN POROUS MEDIUM

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## ABSTRACT

The thermal instability of a rotating Maxwellian viscoelastic fluid permeated with suspended partiles, in porous medium, is considered. The presence of each-rotation, suspended particles and viscoelasticity brings oscillatory modes in the system which were nonexistent in their absence. For stationary convection, the Maxwellian viscoelastic fluid behaves like a Newtonian fluid and the medium permeability succeeds in stabilizing the thermal instability of a fluid-particle layer for certain wave numbers in the presence of rotation which were unstable in the absence of rotation. However, the suspended particles and the rotation are found to have destabilizing and stabilizing effects respectively on the thermal instability problem for stationary convection.

**1. INTRODUCTION.** A detailed account of the thermal instability in Newtonian fluids has been given by Chandrasekhar [1]. Lapwood [2] has studied the stability of convective flow in a porous medium using Rayleigh's procedure. The Rayleigh instability of a thermal boundary layer in flow through a porous medium has been considered by Wooding [3]. When the fluid slowly percolates through the pores of the rock, the gross effect is represented by the well-known Darcy's law.

Chandra [4] observed in an air layer that convection occured at much lower gradients than predicted and appeared as irregular strips of elongated cells with fluid rising at the centre if the layer depth was less than 7 mm and called this motion "columnar instability". However, for layers deeper than 10 mm, a Bénard type cellular convection was observed. Chandra [4] added an aerosol to mark the flow pattern. Thus there is a decades-old contradiction between the theory and the experiment. Scanlon and Segel [5] have considered the effect of suspended particles on the onset of thermal convection and found that the critical Rayleigh number was reduced solely because the heat capacity of the pure fluid was supplemented by that of the particles. The effect of suspended particles was thus found to destablize the layer.

In all the above studies, the fluid has been considered to be Newtonian. The stability of a horizontal layer of Maxwell's viscoelastic fluid heated from below has been investigated by Vest and Arpaci [6]. The nature of the instability and some factors may have different effects on viscoelastic fluids as compared to the Newtonian fluids. For example, Bhatia and Steiner [7] have considered the effect of a uniform rotation on the thermal instability of a Maxwell fluid and have found that rotation has a destabilizing effect in contrast to the stabilizing effect on Newtonian fluid. Experimental demonstration by Toms and strawbridge [8] has revealed that a dilute solution of methyl methacrylate in $n$-butyl acetate agrees well with the the theoretical model of Oldroyd fluid. The thermal instability of an Oldroydian viscoelastic fluid has been considered in the presence of rotation (Eltayeb [9], Sharma [10]).

The present paper attempts to study the thermal instability of rotating Maxwellian viscoelastic fluid, permeated with suspended particles, in porous medium. The knowledge regarding fluid- particle mixtures is not commensurate with their scientific and industrial importance. The analysis would be relevant to the stability of some polymer solutions and the problem finds its usefulness in several geophysical situations (e.g. convection in petroleum deposits below the surface of Earth where the Earth's rotation plays a significant role on convection and in extraction of energy) and in chemical technology. These aspects form the motivation for the present study.

**2. PERTURBATION EQUATIONS.** Consider an infinite horizontal layer of a Maxwellian viscoelastic fluid permeated with suspended particles and bounded by the planes $z = 0$ and $z = d$ in porous medium. This layer is heated from below such that a steady adverse temperature gradient $\beta (= |\frac{dT}{dz}|)$ is maintained. The system is acted on by gravity force $\mathbf{g}(0, 0, -g)$ and uniform rotation $\vec{\Omega}(0, 0, \Omega)$. Let $T_{ij}, z_{ij}, e_{ij}, \delta_{ij}, \mu, \lambda, p, u_i, x_i$ and $\frac{d}{dt}$ denote respectively the total stress tensor, the shear stress tensor, the rate-of-strain tensor, the Kronecker delta, the viscosity, the stress relaxation time, the isotropic pressure, the velocity vector, the position vector and the mobile operator. Then the Maxwellian viscoelastic fluid is described by the constitutive relations

$$T_{ij}=-p\delta_{ij}+z_{ij},$$

$$\left(1+\lambda\frac{d}{dt}\right)z_{ij}=2\mu e_{ij},$$

$$e_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \qquad \text{...(1)}$$

When the fluid slowly percolates through the pores of the rock, the gross effect is represneted by the usual Darcy's law. As a consequence, the resistance term $-\left(\frac{\mu}{k_1}\right)\boldsymbol{v}$ will replace the usual viscous term in the equations of motion. Here $k_1$ is the permeability of the medium and $\boldsymbol{v}$ is the filter velocity of the fluid, the porous medium being homogeneous and isotropic.

Let $\rho, \mu, p, \vec{u}(u, v, w)$ and $\vec{\Omega}(0, 0, \Omega)$ denote respectively, the density, the viscosity, the pressure, the velocity of pure fluid and uniform rotation; $\boldsymbol{v}(\bar{x}, t)$ and $N(\bar{x}, t)$ denote the velocity and the number density of the particles respecitvely. $k = 6\pi\rho\upsilon\eta$, where $\eta$ is the particle radius is the Stoke's drag coefficient $\boldsymbol{v} = (l, r, s)$, $\bar{x} = (x, y, z)$, $\bar{\lambda} = (0, 0, 1)$ and $\varepsilon$ is the medium porosity. Then the equations of motion and continuity for the rotating Maxwellian viscoelastic fluid permeated with suspended particles in a porous medium are

$$\left(1+\lambda\frac{d}{dt}\right)\frac{\rho}{\varepsilon}\left[\frac{\partial \boldsymbol{u}}{\partial t}+\frac{1}{\varepsilon}(\vec{u}.\nabla)\vec{u}\right]=\left(1+\lambda\frac{d}{dt}\right)\left[-\nabla p-\rho g\bar{\lambda}+\frac{kN}{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})+\frac{2\rho}{\varepsilon}(\boldsymbol{u}\times\vec{\Omega})\right]-\frac{\mu}{k_1}\boldsymbol{u} \qquad \text{...(2)}$$

$$\nabla\,.\,\boldsymbol{u}=0. \qquad \text{...(3)}$$

In the equations of motion (2), the presence of partilces adds an extra force term, proportional to the velocity difference between particles and fluid. Since the force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid, there must be an extra force term, equal in magnitude but opposite in sign, in the equations of motion of the particles. The distances between particles are assumed quite large compared with their diameter so that interparticle reactions are ignored. The effects of pressure, gravity, Darcian force and Coriolis force on the suspended particles are negligibly small and therefore ignored. If $mN$ is the mass of particles per unit volume, then the equations of motion and continuity for the particles, under the above assumptions are

$$mN\left[\frac{\partial \boldsymbol{v}}{\partial t}+\frac{1}{\varepsilon}(\boldsymbol{v}\,.\,\nabla)\boldsymbol{v}\right]=kN\,(\boldsymbol{u}-\boldsymbol{v}), \qquad \text{...(4)}$$

$$\varepsilon\frac{\partial N}{\partial t}+\nabla.\,(N\boldsymbol{v})=0. \qquad \text{...(5)}$$

Let $c$, $c_{pt}$, $T$ and $q$ denote respectively the heat capacity of the fluid, the heat capacity of the particles, the temperature and the "effective thermal conductivity" of the pure fluid. When the fluid and the particles are in thermal equilibrium, the equation of heat conduction gives

$$[\rho c\varepsilon+\rho_s c_s(1-\varepsilon)]\frac{\partial T}{\partial t}+\rho c(\boldsymbol{u}.\nabla)T+mNc_{pt}\left(\varepsilon\frac{\partial}{\partial t}+\boldsymbol{v}\,.\,\nabla\right)T$$

$$=q\nabla^2 T, \qquad \text{...(6)}$$

where $\rho_s$, $c_s$ stand for the density and the heat capacity of the solid matrix, respectively.

The initial state of the system is taken to be a quiescent layer (no settling) with a uniform particle distribution $N_0$. The initial state

$$\boldsymbol{u}=0,\ \boldsymbol{v}=0,\ T=-\beta z,\ N_0=\text{constant} \qquad \text{...(7)}$$

is an exact solution to the governing equations. The equatuion of state for the fluid is

$$\rho=\rho_0\,[1-\alpha(T-T_0)], \qquad \text{...(8)}$$

where $\rho_0$, $T_0$ are respectively, the mean density and the temperature of the clean fluid at the bottom surface $z=0$ and $\alpha$ is the coefficient of thermal expansion.

Let $\delta\rho$, $\delta p$, $\theta$, $\boldsymbol{u}(u, v, w)$, $\boldsymbol{v}(l, r, s)$ and $N$ denote respectively perturbations in density $\rho$, pressure $p$, temperature $T$, fluid velocity (zero initially), particle velocity (zero initially) and number denisty $N_0$. Then the linearized perturbation equations of the fluid-particle layer are

$$\left(1+\lambda\frac{d}{dt}\right)\frac{\rho_0}{\varepsilon}\frac{\partial \boldsymbol{u}}{\partial t}=\left(1+\lambda\frac{d}{dt}\right)\left[-\nabla\delta p+g\rho_0\alpha\theta\vec{\lambda}+\frac{kN_0}{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\right.$$

$$+\frac{2\rho_0}{\varepsilon}(\boldsymbol{u}+\vec{\Omega})\Big]-\frac{\mu}{k_1}\boldsymbol{u} \qquad ...(9)$$

$$\nabla\,.\,\boldsymbol{u}=0 \qquad ...(10)$$

$$mN_0\frac{\partial v}{\partial t}=kn_0(\boldsymbol{u}-\boldsymbol{v}), \qquad ...(11)$$

$$\varepsilon\frac{\partial M}{\partial t}+\nabla.\boldsymbol{v}=0 \qquad ...(12)$$

and

$$(E+h\varepsilon)\frac{\partial\theta}{\partial t}+(\boldsymbol{u}.\nabla)T+h(\boldsymbol{v}.\nabla)T=\frac{q}{\rho_0 c}\nabla^2\theta \qquad ...(13)$$

where

$$M=\frac{N}{N_0},\ E=\varepsilon+(1-\varepsilon)\frac{\rho_s c_s}{\rho c},$$

$$h=\frac{mN_0}{\rho_0}\frac{c_{pt}}{c}=f\frac{c_{pt}}{c}\text{ and }f=\frac{mN_0}{\rho_0}.$$

Now introducing the non-dimensional quantities defined by

$$z=z^*d,\ t=\frac{d^2}{x}t^*,\ \boldsymbol{u}=\frac{x}{d}\boldsymbol{u}^*,\ p=\frac{\rho v x}{d^2}p^*,\ \theta=(\beta d)\theta^*,\ \Omega=\frac{x}{d^2}\Omega^* \qquad ..(14)$$

and omitting the asterisks for simplicity, (9)-(13) in the non-dimensional forms become

$$\left(1+\lambda'\frac{d}{dt}\right)N_p^{-1}\frac{\partial\boldsymbol{u}}{\partial t}=\left(1+\lambda'\frac{d}{dt}\right)\Big[-\nabla\delta p+N_R\theta\vec{\lambda}+\omega(\boldsymbol{v}-\boldsymbol{u})$$

$$+2N_p^{-1}(\boldsymbol{u}\times\vec{\Omega})\Big]-p^{-1}\boldsymbol{u}, \qquad ...(15)$$

$$\nabla.\boldsymbol{u}=0, \qquad ...(16)$$

$$\left(z\frac{\partial}{\partial t}+1\right)\vec{v}=\vec{u}, \qquad ...(17)$$

$$(E+h\varepsilon)\frac{\partial\theta}{\partial t}=(w+hs)+\nabla^2\theta, \qquad ...(18)$$

where $N_p=\frac{\varepsilon v}{x}$ is the modified Prandtl number, $N_R=\frac{g\alpha\beta d^4}{vx}$ is the Rayleigh number, $\omega=\frac{kN_0d^2}{\rho_0 v\varepsilon}$, $f=\frac{mN_0}{\rho_0}=zwN_p$ is the mass fraction, $z=\frac{mx}{kd^2}$, $p=\frac{k_1}{d^2}$ and $\lambda'=\frac{\lambda x}{d^2}$. Also $w$, $s$ are the vertical fluid and particle velocities, $v$ is the kinematic viscosity of the fluid and $x$ is the thermal diffusivity.

Let us consider the case of two free surfaces having uniform temperatures. The scaled boundary conditions appropriate for the problem are

$$w = \frac{\partial^2 w}{\partial z^2} = \theta = 0 \text{ at } z = 0 \text{ and } 1. \qquad ...(19)$$

Eliminating $\boldsymbol{v}$ from (15), (17) and (18), the fluid and heat equations become

$$[L_1(1 + \lambda' \frac{d}{dt}) + p^{-1} L_2]\zeta = (1 + \lambda' \frac{d}{dt})(2L_2 N_p^{-1} \Omega D w), \qquad ...(20)$$

$$[L_1(1 + \lambda' \frac{d}{dt}) + p^{-1} L_2]\nabla^2 w + (1 + \lambda' \frac{d}{dt}) L_2 (N_R \nabla_1^2 \theta - 2N_p^{-1} \Omega D\zeta] \qquad ...(21)$$

and $$L_2[c(E + h\varepsilon)\frac{\partial}{dt} - \nabla^2]\theta = [z\frac{\partial}{\partial t} + H]w, \qquad ...(22)$$

where we have taken twice the curl of fluid equations and taking $z$-components only, and $\zeta = (\nabla \times \boldsymbol{u})_z$ is the $z$-components of vorticity

$$L_1 = N_p^{-1}\left(z\frac{\partial^2}{\partial t^2} + F\frac{\partial}{\partial t}\right), L_2 = z\frac{\partial}{\partial t} + 1, F = f + 1,$$

$$H = h + 1, \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Following the normal mode analysis we assume that the solutions of (20)-(22) are given by

$$[w, \zeta, \theta] = [W(z), Z(z), \Theta(z)] \exp(ik_x x + ik_y y + nt), \qquad ...(23)$$

where $n$ is the growth rate and $k = (k_x^2 + k_y^2)^{1/2}$ is the wave number of the disturbance.

Equations (20)-(22), with the help of expression (23), take the forms

$$[(1 + \lambda' n)L_1 + p^{-1} L_2]z = 2(1 + \lambda' n) L_2 N_p^{-1} \Omega D W, \qquad ...(24)$$

$$[(1 + \lambda' n)L_1 + p^{-1} L_2](D^2 - k^2)W = -(1 + \lambda' n) L_2 N_k k^2 \Theta$$
$$- 2(1 + \lambda' n) L_2 N_p^{-1} \Omega\, Dz \qquad ...(25)$$

$$L_2\ [(E - h\varepsilon)n - (d^2 - k^2)\Theta = (zn + H)W \qquad ...(26)$$

Eliminating $z$ and $\Theta$ between (24)-(26), we obtain

$$[(1+\lambda' n)L_1 + p^{-1}L_2]^2\,[(E+h\varepsilon)n - (D^2-k^2)][(D^2-k^2)W$$

$$= -(1+\lambda' n)N_R k^2[(1+\lambda' n)L_1 + p^{-1}L_2](zn+H)W$$

$$-\,4(1+\lambda' n)^2 L_2^2 N_p^{-2}\Omega^2\,[(E+h\varepsilon)n - (D^2-k^2)]D^2W \qquad ...(27)$$

where $L_1 = N_p^{-1}(zn^2+F_n)$, $L_2 = (1+zn)$ and $D = \frac{d}{dz}$.

Here we consider the case of two free boundaries. The dimensionless boundary conditions appropriate for the problem are

$$W = D^2W = Dz = \Theta = 0 \text{ at } z = 0 \text{ and } 1. \qquad ...(28)$$

The solution of (27) satisfying (28) is given by

$$W = W_0 \sin \pi z, \qquad ...(29)$$

where $W_0$ is a contant. Substituting (29) in (27), we obtain the dispersion relations

$$N_R = \frac{[(E+h\varepsilon)n + (\pi^2+k^2)] \times [\{(1+\lambda' n)L_1 + p^{-1}L_2\}^2(\pi^2+k^2) + 4\pi^2L_2^2\Omega^2N_p^{-2}(1+\lambda' n)^2]}{k^2(1+\lambda' n)[(1+\lambda' n)L_1 + p^{-1}L_2][zn+H]} \qquad ...(30)$$

**3. THE STATIONARY CONVECTION.** When the instability sets in as stationary convection, the marginal state will be characterized by $n = 0$. Putting $n = 0$, the dispersion relation (30) reduces to

$$N_R = \frac{(\pi^2+k^2)^2}{Hpk^2} + 4N_p^{-2}\Omega^2\pi^2(\pi^2+k^2)\,\frac{p}{Hk^2} \qquad ...(31)$$

Thus for stationary convection, the stress relaxation time parameter vanishes with $n$ and the Maxwell fluid behaves like a Newtonian fluid.

Equation (31) can be written as

$$N_R = \frac{(\pi^2+k^2)^2}{Hpk^2} + 4\Omega^2\pi^2(\pi^2+k^2)\,\frac{p}{Hk^2N_p^2} \qquad ...(32)$$

To study the effects of suspended particles, medium permeability and rotation, we examine the natures of $\frac{dN_R}{dH}$, $\frac{dN_R}{dP}$ and $\frac{dN_R}{d\Omega}$ respectively. Equation (32) gives

$$\frac{dN_R}{dH} = -\left[ \frac{(\pi^2+k^2)^2}{H^2k^2p} + \frac{4\Omega^2\pi^2(\pi^2+k^2)p}{H^2k^2N_p^2} \right], \qquad ...(33)$$

which is negative. The effect of suspended particles is, thus, destabilizing on thermal instability of fluid-particle layer in the presence of uniform rotation in porous medium.

From (32), it follows that

$$\frac{dN_R}{dp} = \left[ \frac{(\pi^2+k^2)^2}{Hk^2p^2} - \frac{4\Omega^2\pi^2(\pi^2+k^2)}{Hk^2N_p^2} \right], \qquad ...(34)$$

which is, positive if

$$\left( \frac{\pi^2+k^2}{p^2} \right) < \frac{4\Omega^2\pi^2}{N_p^2}$$

and is negative if

$$\left( \frac{\pi^2+k^2}{p^2} \right) > \frac{4\Omega^2\pi^2}{N_p^2}.$$

Thus, the medium permeability has been stabilizing and destabilizing effects depending upon the values of the various parameters. In the absence of rotation, the medium permeability has a destabilizing effect since for this case

$$\frac{dN_R}{dp} = -\frac{(\pi^2+k^2)^2}{Hp^2k^2}, \qquad ...(35)$$

which is always negative. The medium permeability, thus, succeeds in stabilizing the thermal instability of fluid-particle layer for certain wave numbers in the presence of rotation, which were unstable in the absence of rotation.

Equation (32) also yields

$$\frac{dN_R}{d\Omega} = \frac{8\Omega\pi^2\,(\pi^2+k^2)p}{Hk^2N_p^2} \qquad ...(36)$$

which is always positive. This shows that the rotation has a stabilizing effect on the system.

## 4. STABILITY OF THE SYSTEM AND OSCILLATORY MODES

Multiplying equation (25) by $W^*$, the complex conjugate of $W$, integrating over the range of $z$ and making use of (24), (26) together with the boundary conditions (28), we obtain

$$\begin{aligned}&L_2^*[zn^* + H][1 + \lambda' n^*][1 + \lambda' n)L_1 + p^{-1}L_2]I_1 = [(1 + \lambda' n^*)(1 + \lambda' n)\\ &L_2 L_2^{2*} N_R k^2][(E + h\varepsilon)n^* I_2 + I_3] - L_2[zn^* + H][1 + \lambda' n]\\ &[(1 + \lambda' n^*)L_1^* + p^{-1}L_2^*]I_4, \qquad \dots(37)\end{aligned}$$

where

$$\begin{aligned}&I_1 = \int_0^1 (|DW|^2 + k^2|W|^2)\,dz,\\ &I_2 = \int_0^1 |\Theta|^2\,dz,\\ &I_3 = \int_0^1 (|D\Theta|^2 + k^2|\Theta|^2)\,dz,\\ &I_3 = \int_0^1 |z|^2\,dz, \qquad \dots(38)\end{aligned}$$

which are all positive definite. Putting $n = in_0$, where $n_0$ is real, in (37) and equating the imaginary parts on both sides, we obtain

$$\begin{aligned}&n_0\,[n_0^4\{N_p^{-1}z^2\lambda'^2(1 + H - F)\}I_1 + \{N_R k^2(E + h\varepsilon)\lambda'^2 z^2\}I_2\\ &+ \{N_R k^2\lambda'^2 z^3\}I_3 + \{N_p^{-1}z^2\lambda'^2(1 - H - F)\}I_4]\\ &+ n_0^2\,[\{H\lambda' + Hz + z) < N_p^{-1}\,(z + \lambda' F) > - (H\lambda' + \lambda' + z)\\ &< N_p^{-1}\,Fz + z^2p^{-1} > - N_p^{-1}\lambda' zH + z^2\lambda' p^{-1}\}I_1\\ &+ \{N_R k^2\,(E + h\varepsilon)(z^2 + \lambda'^2)\}I_2 + \{N_R k^2\,(z^3 + \lambda'^2 z)\}I_3\\ &+ \{N_p^{-1}z^2\,(1 - F - H) - N_p^{-1}\,HF\lambda'^2 + Hz^2\lambda' p^{-1} - z^3p^{-1}\}I_4]\\ &+ \{N_R k^2 z\}I_3 + \{H\lambda' p^{-1} - N_p^{-1}HF - zp^{-1}\}\,I_4] = 0. \qquad \dots(39)\end{aligned}$$

Equation (39) implies that $n_0 = 0$ or $n_0 \neq 0$, which means that the modes may be oscillatory or non-oscillatory. In the absence of rotation, suspended particles and for Newtonian fluid ($\lambda = 0$), (39) reduces to

$$n_0\left[\frac{I_1}{N_p} + N_k k^2 E I_2\right] = 0, \qquad \dots(40)$$

and terms in brackets are positive definite. Thus $n_0 = 0$, which means that oscillatory modes are not allowed and the principle of exchange of stabilities is satisfied for a porous medium in the absence of rotation, suspended particles and for Newtonian fluid. This result is true for the porous as well as non-porous (Chandrasekhar [1]) medium. The presence of each-rotation, suspended particles and viscoelasticity, brings oscillatory modes (as $n_0$ may not be zero) which were non-existent in their absence.

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## REFERENCES

[1] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability,* Dover Publications, New York 1981.
[2] E.R. Lapwood, *Proc. Camb. Philos. Soc.,* **44**, (1948) 508.
[3] R.A. Wooding, *J. Fluid Mech.,* **9**, (1960) 183.
[4] K. Chandra, *Proc. Roy. Soc. (London), A* **164** (1938), 231.
[5] J.W. Scanlon and L.A. Segel, *Phys. Fluids,* **16** (1973) 1573.
[6] C.M. Vest and V.S. Arpaci, *J. Fluid Mech.,* **36** (1969) 613.
[7] P.K. Bhatia and J.M. Steiner, *Z. Angew. Math. Mech.,* **52** (1972), 321.
[8] B.A. Toms and D.J. Stawbridge, *Trans. Faraday Soc.,* **49** (1953) ,225.
[9] I.A. Eltayeb, *Z. Angew Math. Mech.,* **55** (1975), 599.
[10] R.C. Sharma, *Acta Phys. Hung.,* **40** (1976), 11..

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# SPECIAL, UNION AND HYPERASYMPTOTIC CURVES OF A TACHIBANA HYPERSURFACE

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## ABSTRACT

Special, union and hyperasymptotic curves of a Riemannian hypersurface have been studied by Singh [1]. The object of this paper is to be investigate these curves in a Tachibana hypersurface.

**1. INTRODUCTION**. In an $(n+1)$ dimensional complex space $C_{n+1}$ referred to an allowable coordinate system

$$(z^i, z^{\bar{i}}) \equiv (z^1, z^2, \ldots, z^{n+1}, z^{\bar{1}}, z^{\bar{2}}, \ldots, z^{\overline{n+1}})^* ,$$

let us introduce the metric defined by the positive Hermitian form

$$ds^2 = 2g_{i\bar{j}}(z^i, z^{\bar{i}})\, dz^i\, dz^{\bar{j}} . \qquad \ldots (1.1)$$

If the tensor $g_{i\bar{j}}$ also satisfies the Kaehler's condition

$$\frac{\partial g_{i\bar{j}}}{\partial z^{k}} = \frac{\partial g_{i\bar{k}}}{\partial z^{\bar{j}}} , \qquad \ldots (1.2)$$

then the complex space with metric satisfying the condition (1.2) is called Kaehler space. We always assume the self adjointness of the indices [2].

If a Kaehler space satisfies the conditions

$$F^{h}_{j,\,i} = 0,\; F^{\bar{h}}_{\bar{j},\,\bar{i}} = 0, \qquad \ldots (1.3a)$$

$$F_i = -F^{k}_{i,\,j} = 0,\; F_{\bar{i}} = -F^{\bar{k}}_{\bar{i},\,\bar{j}} = 0. \qquad \ldots (1.3b)$$

where the comma (,) followed by an index denotes the covarient differentiation w.r.t. the Christoffel's symbols formed with metric tensor, then the space is called an $(n+1)$ dimensional Tachibana space and is denoted by $T^{c}_{n+1}$.

The angle θ between two self adjoint vectors $\bar{u}$ and $\bar{v}$, whose components are $(r^i, r^{\bar{i}})$ and $(s^i, s^{\bar{i}})$ is given by

$$\cos\theta = \frac{g_{i\bar{j}}\, r^i s^{\bar{j}} + g_{ij}\, r^i s^j}{2\sqrt{(g_{i\bar{j}}\, r^i r^j)}\,\sqrt{(g_{\bar{i}\bar{j}}\, s^i s^j)}} \qquad \dots (1.4)$$

Let us consider an analytic hypersurface $T^c_n$ of $T^c_{n+1}$. If $(u^\alpha, u^{\bar{\alpha}}) \equiv (u^1, \dots, u^n, u^{\bar{1}}, \dots, u^{\bar{n}})$ denote the coordinates of a point in $T^c_n$, the equations of the analytic hypersurface $T^c_n$ may be written in the form

$$z^i = z^i(u^\alpha),\ z^{\bar{i}} = z^{\bar{i}}(u^{\bar{\alpha}}) \qquad \dots (1.5)$$

We quote below some fundamental formulae from [3] which will be used in the later part of this paper. Suppose that $g_{\alpha\beta}$ is the fundamental metric tensor of $T^c_n$, then we have

$$g_{\alpha\beta} = g_{i\bar{j}}\, B^i_\alpha\, B^{\bar{j}}_{\bar{\beta}} \qquad \dots (1.6)$$

where $B^i_\alpha = \dfrac{\partial z^i}{\partial u^\alpha}$, $B^{\bar{i}}_{\bar{\beta}} = \dfrac{\partial z^{\bar{i}}}{\partial u^{\bar{\beta}}}$.

If $(N^i, N^{\bar{i}})$ be the components of unit normal vector to the hypersurface, then

$$2 g_{i\bar{j}}\, N^i N^{\bar{j}} = 1 \qquad \dots (1.7)$$

and $$g_{i\bar{j}}\, N^i B^{\bar{j}}_{\bar{\beta}} = 0,\ g_{\bar{i}j} N^{\bar{j}} B^i_\alpha = 0\,. \qquad \dots (1.8)$$

The unit vector $(\xi^i, \xi^{\bar{i}})$ orthogonal to $\left(\dfrac{dz^i}{ds}, \dfrac{dz^{\bar{i}}}{ds}\right)$ is given by

$$g_{i\bar{j}} \frac{dz^i}{ds} \xi^{\bar{j}} + g_{\bar{i}j} \frac{dz^{\bar{i}}}{ds} \xi^j = 0 \qquad \dots (1.9)$$

and $$2 g_{i\bar{j}}\, \xi^i \xi^{\bar{j}} = 1\,.$$

Consider a curve $C : z^i = z^i(s),\ z^{\bar{i}} = z^{\bar{i}}(s)$ (where $s$ is real) of $T^c_n$. The components $\dfrac{dz^i}{ds}\left(\text{or } \dfrac{dz^{\bar{i}}}{ds}\right)$ and $\dfrac{du^\alpha}{ds}\left(\text{or } \dfrac{du^{\bar{\alpha}}}{ds}\right)$ of the unit tangent vectors of $C$ with respect to the enveloping space and the hypersurface are related by

$$\frac{dz^i}{ds} = B^i_\alpha \frac{du^\alpha}{ds}, \qquad \dots (1.11a)$$

$$\frac{dz^{\bar{i}}}{ds} = B^{\bar{i}}_{\bar{\alpha}} \frac{du^{\bar{\alpha}}}{ds} \qquad \dots (1.11b)$$

If $(q^i, q^{\bar{i}})$ and $(p^\alpha, p^{\bar{\alpha}})$ are the components of the first curvature vectors with respect to $T^c_{n+1}$ and $T^c_n$ respectively, then

$$q^i = B^i_\alpha p^\alpha + K_n N^i \quad \text{... (1.12a)}$$

$$q^{\bar{i}} = B^{\bar{i}}_{\bar{\alpha}} p^{\bar{\alpha}} + \bar{K}_n N^{\bar{i}} . \quad \text{... (1.12b)}$$

where $K_n$ is the component of the normal curvature of $T^c_n$ in the direction of the curve $C$ and $\bar{K}_n$ is its complex conjugate, and

$$\left.\begin{array}{lll} q^i = dz^i/ds, & \beta\, du^\beta/ds + dz^i/ds, & \bar{\beta}\, du^\beta/ds \\ q^{\bar{i}} = dz^{\bar{i}}/ds, & \beta\, du^\beta/ds + dz^{\bar{i}}/ds, & \bar{\beta}\, du^\beta/ds \\ p^\alpha = du^\alpha/ds, & \beta\, du^\beta/ds + du^\alpha/ds, & \bar{\beta}\, du^\beta/ds \\ p^{\bar{\alpha}} = du^{\bar{\alpha}}/ds, & \beta\, du^\beta/ds + du^{\bar{\alpha}}/ds, & \bar{\beta}\, du^{\bar{\beta}}/ds \end{array}\right\} \text{... (1.13)}$$

$$B^i_{\alpha,\beta} = \Omega_{\alpha\beta} N^i, \qquad B^{\bar{i}}_{\bar{\alpha},\bar{\beta}} = \Omega_{\bar{\alpha}\bar{\beta}} N^{\bar{i}} .$$

$$K_n = \Omega_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds}, \qquad \bar{K}_n = \Omega_{\bar{\alpha}\bar{\beta}} \frac{du^{\bar{\alpha}}}{ds} \frac{du^{\bar{\beta}}}{ds}.$$

$\Omega_{\alpha\beta}$ and $\Omega_{\bar{\alpha}\bar{\beta}}$ are the components of the second fundamental tensors of the hypersurface, $(q^i, q^{\bar{i}})$ and $(p^\alpha, p^{\bar{\alpha}})$ are the components of the first curvature vector of the curve with respect to $T^c_{n+1}$ and $T^c_n$ respectively.

**2. Union and Special Curves in $T^c_n$.** The first two of Frenet's formulae of a curve in $T^c_{n+1}$ are given by

$$\frac{\delta\, \eta^i_{(0)}}{\delta s} = K_{(1)}\, \eta^i_{(1)} \text{ and } \frac{\delta \eta^i_{(1)}}{\delta s} = -K_{(1)}\, \eta^i_{(0)} + K_{(2)}\, \eta^i_{(2)} \quad \text{... (2.1)}$$

and their conjugates, where

$$(\eta^i_{(0)}, \eta^{\bar{i}}_{(0)}) = (dz^i/ds, dz^{\bar{i}}/ds), (\eta^i_{(1)}, \eta^{\bar{i}}_{(1)}) \text{ and } (\eta^i_{(2)}, \eta^{\bar{i}}_{(2)})$$

are the components of unit tangent, unit principal normal vector and unit first binormal vector, $K_{(1)}$ and $K_{(2)}$ are the first and second curvatures of the curve in the embedding space, $\delta/\delta s$ is the usual covariant differentiation.

Consider two congruences of curves $\bar{\lambda}$ and $\bar{\mu}$ given at any point of $T^c_n$ by

$$\lambda^i = \gamma^\alpha B^i_\alpha + cN^i \quad \text{... (2.2a)}$$

its conjugate,

$$\mu^i = s^\alpha B^i_\alpha + DN^i \quad \text{... (2.2b)}$$

and its conjugate.

Let $\Gamma$ be a special curve relative to the congruence $\bar{\lambda}$ and an union curve relative to the congruence $\bar{\mu}$. A curve of the congruence is said to be a special curve relative to the congruence $\bar{\lambda}$ if the vector $(\lambda^i, \lambda^{\bar{i}})$ lies in a variety spanned by the vectors $(p^{\alpha}B^i_{\alpha}, p^{\bar{\alpha}}B^{\bar{i}}_{\bar{\alpha}})$ and $(q^i, q^{\bar{i}})$ and a curve of the congruence is said to be union curve relative to the congruence $\bar{\mu}$ if the vector $(\mu^i, \mu^{\bar{i}})$ lies in a variety spanned by the vectors $(dz^i/ds, dz^{\bar{i}}/ds)$ and $(q^i, q^{\bar{i}})$. In other words

$$(\lambda^i, \lambda^{\bar{i}}) = u(p^{\alpha}B^i_{\alpha}, p^{\bar{\alpha}}B^{\bar{i}}_{\bar{\alpha}}) + w(q^i, q^{\bar{i}})$$

and
$$(\mu^i, \mu^{\bar{i}}) = x(dz^i/ds, dz^{\bar{i}}/ds) + y(q^i, q^{\bar{i}})$$

which yield

$$\lambda^i = up^{\alpha}B^i_{\alpha} + wq^i \text{ and its conjugate,} \qquad \text{... (2.3a)}$$

$$\mu^i = x\,dz^i/ds + yq^i \text{ and its conjugate.} \qquad \text{... (2.3b)}$$

From equations (1.12a), (2.2a), (2.3a), (2.3b) and their conjugates one obtains

$$r^{\alpha} = (u+w)\,p^{\alpha},\; C = wK_n \qquad \text{... (2.4)}$$

$$s^{\alpha} = x\,dx^{\alpha}/ds + yp^{\alpha},\; D = yK_n \qquad \text{... (2.5)}$$

and their conjugate relations.

We define

$$R^2 = 2g_{\alpha\bar{\beta}}\,\gamma^{\alpha}\gamma^{\bar{\beta}},\; S^2 = 2g_{\alpha\bar{\beta}}\,s^{\alpha}s^{\bar{\beta}},\; k^2_{(1)} = 2g_{\alpha\bar{\beta}}\,p^{\alpha}p^{\bar{\beta}} \qquad \text{... (2.6)}$$

where $k_{(1)}$ is the first curvature of the curve in $T^c_n$. Using (1.9), (2.4), its conjugate, (2.5) and its conjugate, we obtain from these equations

$$K_n\,S\cos\theta = Dk_{(1)} \qquad \text{... (2.7)}$$

and its conjugate.

From this equation and the equations

$$1 = 2g_{i\bar{j}}\,\mu^i\mu^{\bar{j}} = S^2 + D\bar{D} \qquad \text{... (2.8)}$$

$$K^2_{(1)} = 2g_{i\bar{j}}\,q^i q^{\bar{j}} \qquad \text{... (2.9)}$$

where
$$K^2_{(1)} = k^2_{(1)} + K_n\bar{K}_n$$

We obtain

$$|K_n| = \{eK_{(1)}(1-S^2)^{1/2}\}/(1-S^2\sin^2\theta)^{1/2} \qquad \text{... (2.10)}$$

and
$$k_{(1)} = \{eK_{(1)}\,S\cos\theta\}/(1-S^2\sin^2\theta)^{1/2} \qquad \text{... (2.11)}$$

where $e = \pm 1$ in order that $e\cos\theta$ be non-negative.

**Theorem 2.1 .** If a special curve relative to a fixed congruence $\overline{\lambda}$ is an union curve relative to another fixed congruence $\overline{\mu}$, then the modulus of the normal and the first curvatures with respect to $T_n^c$ at a given point of the curve are proportional to its first curvature with respect to $T_{n+1}^c$.

**Theorem 2.2 .** If the components of the vector fields $\overline{\lambda}$ and $\overline{\mu}$ tangent to the hypersurface are in the same direction, then the ratio of the two first curvatures is equal to the magnitude of the tangential (to the hypersurface) component of $\overline{\mu}$.

**Proof .** $\theta = 0$ proves both the theorems.

**3. HYPERASYMPTOTIC CURVES IN $T_n^c$ .** A curve of the hypersurface is said to be an hyperasymptotic curve relative to the congruence $\overline{\mu}$, if the vector $(\mu^i, \mu^{\overline{i}})$ lies in the variety spanned by the vectors $(\eta^i_{(0)}, \eta^{\overline{i}}_{(0)})$ and $(\eta^i_{(2)}, \eta^{\overline{i}}_{(2)})$.

In other words

$$(\mu^i, \mu^{\overline{i}}) = X_{(1)}\,(\eta^i_{(0)}, \eta^{\overline{i}}_{(0)}) + Y_{(1)}\,(\eta^i_{(2)}, \eta^{\overline{i}}_{(2)})$$

which yields

$$\mu^i = X_{(1)}\,\eta^i_{(0)} + Y_{(1)}\eta^i_{(2)} \text{ and its conjugate.} \qquad \dots (3.1)$$

From the equation (2.1) and its conjugate we deduce

$$\frac{\delta q^i}{\delta s} = -K^2_{(1)}\,\eta^i_{(1)} + \frac{d}{ds}\log K_{(1)}\,q^i + K_{(1)}\,K_{(2)}\,\eta^i_{(2)} \qquad \dots (3.2)$$

and its conjugate.

Another expression for $\left(\dfrac{\delta q^i}{\delta s}, \dfrac{\delta q^{\overline{i}}}{\delta s}\right)$ will be obtained from the equation (1.12 a, b) after using

$$\frac{\delta N^i}{\delta s} = -\frac{1}{2}\,\Omega_{\overline{\beta}\,\overline{\gamma}}\,g^{\beta\alpha}\,B^i_\alpha\,\frac{dx^{\overline{\gamma}}}{ds} \text{ and its conjugate.} \qquad \dots (3.3)$$

Equations (1.12a), (2.1), (3.1), (3.2) and their conjugates yield

$$s^\alpha = X_{(1)}\,\frac{dx^\alpha}{ds} + z\left(\frac{\delta p^\alpha}{\delta s} - K_{n/2}\,\Omega_{\overline{\beta}\,\overline{\gamma}}\,g^{\overline{\beta}\alpha}\,\frac{dx^{\overline{\gamma}}}{ds} - p^\alpha\,\frac{d}{ds}\log K_{(1)} + K^2_{(1)}\,\frac{dx^\alpha}{ds}\right) \qquad \dots (3.4)$$

$$D = z\left( \Omega_{\alpha\beta}\, p^{\alpha} \frac{dx^{\beta}}{ds} + \frac{d}{ds} K_n - K_n \frac{d}{ds} \log k_{(1)} \right) \qquad \ldots (3.5)$$

and their conjugates, where $z = Y_{(1)}/K_{(1)}\, K_{(2)}$.

Let $(\xi^{\alpha}_{(0)}, \xi^{\alpha}_{(0)}, \xi^{\alpha}_{(1)}, \xi^{\bar{\alpha}}_{(1)})$ and $(\xi^{\alpha}_{(2)}, \xi^{\bar{\alpha}}_{(2)})$ be the unit tangent, unit principal, normal, unit first binormal vectors, $k_{(1)}$ and $k_{(2)}$ be the first and second curvatures of the curve with respect to the hypersurface. We obtain from Frenet's formulae with respect to $T^{c}_{n}$.

$$\frac{\delta p^{\alpha}}{\delta s} = -k^{2}_{(1)} \frac{dx^{\alpha}}{ds} + p^{\alpha} \frac{d}{ds} \log k_{(1)} + k_{(1)} k_{(2)}\, \xi^{\alpha}_{(2)} \qquad \ldots (3.6)$$

and its conjugate.

Equations (3.4), (3.5), (3.6) their conjugates and the definition

$$\cos\phi \left\{ q_{\alpha\bar{\beta}}\, s^{\alpha} \frac{dx^{\bar{\beta}}}{ds} + g_{\bar{\alpha}\beta}\, s^{\bar{\alpha}} \frac{dx^{\beta}}{ds} \right\} \quad S$$

yield

$$S \cos\phi = X(1) \qquad \ldots (3.7)$$

and

$$\left( s^{\alpha} - S\cos\phi \frac{dx^{\alpha}}{ds} \right)\left[ \Omega_{\gamma\beta}\, p^{\gamma} \frac{dx^{\beta}}{ds} + \frac{d}{ds} K_n - K_n \frac{d}{ds} \log k_{(2)} \right]$$

$$= D\left[ K_n \bar{K}_n \frac{dX^{\alpha}}{ds} + p^{\alpha} \frac{d}{ds} \log \frac{k_{(1)}}{K_{(1)}} + k_{(1)}\, k_{(2)}\, \xi^{\alpha}_{(2)} - K_n / 2\, \Omega_{\bar{\beta}\bar{\gamma}}\, g^{\overline{\alpha\beta}} \frac{dx^{\bar{\gamma}}}{ds} \right]$$

and its conjugate.

A hyperasymptotic curve relative to $\bar{\mu}$ is characterised by this equation.

## REFERENCES

[1] U.P. Singh, Special union and hyperasymptotic curves of a Riemannian hypersurface. *Tensor (N.S.)*, **22** (1971), 15-18.

[2] S. Bochner, On compact complex manifolds, *Jour. Ind. Math. Soc.* **11** (1947), 1-21.

[3] K. Yano, *Differential Geometry on Complex and Almost Complex Spaces*. Pergamon Press (1965).

[4] S.C. Saxena and Ram Behari, Hypersurface of a Kaehler manifold, *Proc. Nat. Acad. Sci.* **1**, **28A** (1956), 170-180.

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# SOME DOUBLE INTEGRALS AND LAPLACE TRANSFORMS ASSOCIATED WITH THEM

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## ABSTRACT

In the present paper, we derive two infinite double integrals involving the exponential function, the product of general polynomials $S_V^U[x]$, the polynomial set $S_n^{\alpha,\beta,0}[x]$ and the $H$-function of two variables. Due to the presence of generalized polynomials and functions, the results obtained here are quite general in nature and are capable of yielding a large number of known and new integrals and double Laplace transform of certain functions. To illustrate we have obtained here two particular cases of our result. One of them gives the double Laplace transform of product of two generalizations of Hermite polynomials given by Gould and Hopper [4] and the other is an integral involving the product of Laguerre polynomials $L_n^{(\alpha)}(x)$ and a generalization of Laguerre polynomials $L_n^{(\alpha)}(x, r, \beta)$ introduced by Singh and Srivastava [11].

## 1. INTRODUCTION

(i) The two dimensional Laplace transform of a function $f(x_1, x_2)$ will be defined as

$$L\{f(x_1, x_2) : s_1, s_2\} = \int_0^\infty \int_0^\infty e^{-s_1 x_1 - s_2 x_2} f(x_1, x_2)\, dx_1\, dx_2 \qquad \text{... (1.1)}$$

(ii) The general class of polynomials $S_V^U[x]$ introduced by Srivastava [12, p. 1, eq. (1)]

$$S_V^U[x] = \sum_{K=0}^{[V/U]} \frac{(-V)_{UK} A_{V,K}\, x^K}{K!} \quad (K = 0, 1, 2, \ldots) \qquad \text{... (1.2)}$$

where $U$ is an arbitrary positive integer and the coefficients $A_{V,K}$ $(V, K \geq 0)$ are arbitrary constants, real or complex. By suitably specializing the coefficients $A_{V,K}$, the polynomials $S_V^U[x]$ can easily be reduced to the classical orthogonal polynomials including for example, the Laguerre polynomials $L_n^{(\alpha)}(x)$, the generalized Hermite polynomials $g_V^U(x, h)$ and several others (see, for details [14, p.158-161]).

(iii) The generalized polynomial set $S_n^{\alpha,\beta,\tau}[x]$ is defined by following Rodrigues type formula [9,p.64,eq.(2.1.8)]

$$S_n^{\alpha,\beta,\tau}[x : r, s, q, A, B, m, k, l]$$
$$= (Ax+B)^{-\alpha}(1-\tau x^r)^{-\beta/\tau}\, T_{k,l}^{m+n}\left[(Ax+B)^{\alpha+qn}(1-\tau x^r)^{\beta/\tau+sn}\right] \qquad \dots (1.3)$$

with the differential operator $T_{K,l}$ being defined as

$$T_{k,l} \equiv x^l\left(k + x\frac{d}{dx}\right) \qquad \dots (1.4)$$

The explicit form of this generalization polynomial set [9,p.71, eq.(2.3.4)] is given by

$$S_n^{\alpha,\beta,\tau}[x : r, s, q, A, B, m, k, l] = B^{qn}\, x^{l(m+n)}\, (1-\tau x^r)^{sn}\, l^{m+n}$$
$$\sum_{v=0}^{m+n}\sum_{u=0}^{v}\sum_{j=0}^{m+n}\sum_{i=0}^{j} \frac{(-1)^j(-j)_i(\alpha)_j(-v)_u(-\alpha-qn)_i}{u!\,v!\,i!\,j!\,(1-\alpha-j)_i}$$
$$\left(-\frac{\beta}{\tau}-sn\right)_v\left(\frac{i+k+ru}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^v\left(\frac{Ax}{B}\right)^i \qquad \dots (1.5)$$

It may be pointed out here that the polynomial set defined by (1.2) is very general in nature and it unifies and extends a number of classical polynomials introduced and studied by various research workers such as Chatterjea [1], Dhillon [3], Gould- Hopper [4], Krall and Frink [7], Singh [10] and Singh and Srivastava [11] etc. Some of the special cases of (1.4) are given by Raizada in a tabular form [9,p.65]. We shall require the following special case of (1.4).

If we take $A = 1$, $B = 0$ and let $\tau \to 0$ in (1.4) and use the well known confluence principle

$$\lim_{|b|\to\infty} (b)_n (x/b)^n = x^n \qquad \dots (1.6)$$

therein, we arrive at the following polynomial set

$$S_n^{\alpha,\beta,0}[x] = S_n^{\alpha,\beta,0}[x : r, q, l, 0, m, k, l]$$

$$= x^{an+l(m+n)} l^{m+n} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \frac{(-v)_u \left(\frac{\alpha+an+k+ru}{l}\right)_{m+n} (\beta x^r)^v}{u!\, v!} \quad \dots (1.7)$$

(iv) The $H$-function of two variables defined by Mittal and Gupta [8,p.117, eq.(1.1.1)], possesses the following integral representation

$$H[x, y] = H^{O, N : M_1, N_1 : M_2, N_2}_{P, Q : P_1, Q_1 : P_2, Q_2}\left[\begin{matrix} x \\ y \end{matrix} \middle| \begin{matrix} (a_j : \alpha_j', \alpha_j'')_{1,p} : (c_j', \gamma_j')_{1,P_1} : (c_j'', \gamma_j'')_{1,P_2} \\ (b_j : \beta_j', \beta_j'')_{1,Q} : (d_j', \delta_j')_{1,Q_1} : (d_j'', \delta_j'')_{1,Q_2} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi(\xi, \eta)\ \phi_1(\xi)\, \phi_2(\eta)\, x^{\xi} y^{\eta}\, d\xi\, d\eta,\ \omega = \sqrt{-1} \quad \dots (1.8)$$

where

$$\phi(\xi, \eta) = \frac{\prod_{j=1}^{N} \Gamma\left(1 - a_j + \alpha_j'\xi + \alpha_j''\eta\right)}{\prod_{j=N+1}^{p} \Gamma\left(a_j - \alpha_j'\xi + \alpha_j''\eta\right) \prod_{j=1}^{Q} \Gamma\left(1 - b_j + \beta_j'\xi + \beta_j''\eta\right)} \quad \dots (1.9)$$

$$\phi_1(\xi) = \frac{\prod_{j=1}^{M_1} \Gamma\left(d_j' - \delta_j'\xi\right) \prod_{j=1}^{N_1} \Gamma\left(1 - c_j' + \gamma_j'\xi\right)}{\prod_{j=M_1+1}^{Q_1} \Gamma\left(1 - d_j' + \delta_j'\xi\right) \prod_{j=N_1+1}^{P_1} \Gamma\left(c_j' - \gamma_j'\xi\right)} \quad \dots (1.10)$$

and
$$\phi_2(\eta) = \frac{\prod_{j=1}^{M_2} \Gamma\left(d_j'' - \delta_j''\right)\eta \prod_{j=1}^{N_2} \Gamma\left(1 - c_j'' + \gamma_j''\eta\right)}{\prod_{j=M_2+1}^{Q_2} \Gamma\left(1 - d_j'' + \delta_j''\eta\right) \prod_{j=N_2+1}^{P_2} \Gamma\left(c_j'' - \gamma_j''\eta\right)} \quad \dots (1.11)$$

For the convergence, existence conditions and other details of the $H$-function of two variables, we refer to the book [13,p.83, eq.(6.1.5), (6.1.6)].

## 2. FIRST INTEGRAL

$$\int_0^\infty \int_0^\infty (\lambda_1 x_1 + \lambda_2 x_2)^{\sigma_1 - 1} (\mu_1 x_1 + \mu_2 x_2)^{\sigma_2 - 1} \exp\left[-s_1(\lambda_1 x_1 + \lambda_2 x_2) - s_2(\mu_1 x_1 + \mu_2 x_2)\right]$$
$$S_V^U[\lambda_1 x_1 + \lambda_2 x_2]\ S_n^{\alpha,\beta,0}[\mu_1 x_1 + \mu_2 x_2]\ dx_1\, dx_2$$

$$= \frac{1}{R} \sum_{K=0}^{[V/U]} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \theta(K, u, v) \frac{\Gamma(\sigma_1 + K)\, \Gamma(\sigma_2 + qn + l(m+n) + vr)}{s_1^{\sigma_1 + k}\, s_2^{\sigma_2 + qn + (m+n) + vr}} \quad \text{... (2.1)}$$

where

$$\theta(K, u, v) = \frac{(-V)_{UK} A_{V, K}\, l^{m+n} (-v)_u\, \beta^v}{K!\, u!\, v!} \left(\frac{\alpha + qn + k + ru}{l}\right)_{m+n} \quad \text{... (2.2)}$$

and

$$R = \begin{vmatrix} \lambda_1, \mu_1 \\ \lambda_2, \mu_2 \end{vmatrix} \neq 0, \ \mathrm{Re}(\sigma_1) > 0, \ \mathrm{Re}(s_i) > 0 \ (i = 1, 2,) \quad \text{... (2.3)}$$

**Proof** : To prove (2.1), we make use of the known double integral [15,p.241, eq.(7)]

$$\int_0^\infty \int_0^\infty F(\lambda_1 x_1 + \lambda_2 x_2, \mu_1 x_1 + \mu_2 x_2)\, dx_1\, dx_2 = \frac{1}{R} \int_0^\infty \int_0^\infty F(u_1, u_2)\, du_1\, du_2 \quad \text{... (2.4)}$$

where $R$ is defined as in (2.3).

If we take

$$F(\lambda_1 x_1 + \lambda_2 x_2, \mu_1 x_1 + \mu_2 x_2) = f_1(\lambda_1 x_1 + \lambda_2 x_2)\, f_2(\mu_1 x_1 + \mu_2 x_2)$$

then $\int_0^\infty \int_0^\infty f_1(\lambda_1 x_1 + \lambda_2 x_2)\, f_2(\mu_1 x_1 + \mu_2 x_2)\, dx_1\, dx_2$

$$= \frac{1}{R} \int_0^\infty f_1(u_1)\, dx_1 \int_0^\infty f_2(u_2) du_2 \quad \text{... (2.5)}$$

Consider

$$f_1(\lambda_1 x_1 + \lambda_2 x_2) = (\lambda_1 x_1 + \lambda_2 x_2)^{\sigma_1 - 1} \exp[-s_1(\lambda_1 x_1 + \lambda_2 x_2)]\, S_V^U[\lambda_1 x_1 + \lambda_2 x_2]$$

and

$$f_2(\mu_1 x_1 + \mu_2 x_2) = (\mu_1 x_1 + \mu_2 x_2)^{\sigma_2 - 1} \exp[-s_2(\mu_1 x_1 + \mu_2 x_2)]\, S_n^{\alpha, \beta, 0}[\mu_1 x_1 + \mu_2 x_2]$$

Then, from (2.5), we have

$$\int_0^\infty \int_0^\infty (\lambda_1 x_1 + \lambda_2 x_2)^{\sigma_1 - 1} (\mu_1 x_1 + \mu_2 x_2)^{\sigma_2 - 1}$$
$$\exp[-s_1(\lambda_1 x_1 + \lambda_2 x_2) - s_2(\mu_1 x_1 + \mu_2 x_2)]$$
$$S_V^U[\lambda_1 x_1 + \lambda_2 x_2]\ S_n^{\alpha, \beta, 0}[\mu_1 x_1 + \mu_2 x_2]\ dx_1\, dx_2$$
$$= \frac{1}{R} \int_0^\infty u_1^{\sigma_1 - 1} e^{-s_1 u_1} S_V^U[u_1]\, du_1 \int_0^\infty u_2^{\sigma_2 - 1} e^{-s_2 u_2}\ S_n^{\alpha, \beta, 0}[u_2]\, du_2 \quad \text{... (2.6)}$$

On expressing the general class of polynomials $S_V^U[u_1]$ and generalized polynomial set $S_n^{\alpha, \beta, 0}[\mu_2]$ occurring on the right hand side of (2.6) in series form with the help of (1.2) and (1.7) respectively, interchanging the order of integrals and summation in the result thus obtained and evaluating the $u_1$ and $u_2$ integrals with the help of a known formula [5,p.317,eqn.(3.381.4)], we arrive at the desired result.

### 3. SECOND INTEGRAL

$$\int_0^\infty \int_0^\infty (\lambda_1 x_1 + \lambda_2 x_2)^{\sigma_1 - 1} (\mu_1 x_1 + \mu_2 x_2)^{\sigma_2 - 1}$$

$$\exp[-s_1(\lambda_1 x_1 + \lambda_2 x_2) - s_2(\mu_1 x_1 + \mu_2 x_2)]\, S_V^U [\lambda_1 x_1 + \lambda_2 x_2]$$

$$S_n^{\alpha,\beta,0} [\mu_1 x_1 + \mu_2 x_2]\, H^{O,N:M_1,N_1:M_2,N_2}_{P,Q:P_1,Q_1:P_2,Q_2} \left[ \begin{array}{c} z_1 (\lambda_1 x_1 + \lambda_2 x_2)^{\rho_1} \\ z_2 (\mu_1 x_1 + \mu_2 x_2)^{\rho_2} \end{array} \right|$$

$$\left. \begin{array}{l} (a_j : \alpha_j', \alpha_j'')_{1,P} : (c_j', \gamma_j')_{1,P_1} : (c_j'', \gamma_j'')_{1,P_2} \\ (b_j : \beta_j', \beta_j'')_{1,Q} : (d_j', \delta_j')_{1,Q_1} : (d_j'', \delta_j'')_{1,Q_2} \end{array} \right] dx_1\, dx_2$$

$$= \frac{1}{R} \sum_{K=0}^{[V/U]} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \frac{\theta(K,u,v)}{s_1^{\sigma_1 + K}\, s_2^{\sigma_2 + qn + l(m+n) + vr}} H^{O,N:M_1,N_1+1:M_2,N_2+1}_{P,Q:P_1+1,Q_1:P_2+1,Q_2}$$

$$\left[ \begin{array}{c} z_1 s_1^{-\rho_1} \\ z_2 s_2^{-\rho_2} \end{array} \right| \begin{array}{l} (a_j : \alpha_j', \alpha_j'')_{1,P} : (1 - \sigma_1 - K, \rho_1), (c_j', \gamma_j')_{1,P_1}; \\ (b_j : \beta_j', \beta_j'')_{1,Q} : (d_j', \delta_j')_{1,Q_1}; \end{array}$$

$$\left. \begin{array}{c} (1 - \sigma_2 - qn - l(m+n) - vr, \rho_2), (c_j'', \gamma_j'')_{1,P_2} \\ (d_j'', \delta_j'')_{1,Q_2} \end{array} \right] \quad \ldots (3.1)$$

where $\theta$ and $R$ are defined as in (2.2) and (2.3) respectively and the following conditions are satisfied :

(i) $\rho_i \geq 0$ ; $\mathrm{Re}(s_1) > 0$, $(i = 1, 2)$

(ii) $\mathrm{Re}(\sigma_i) + \rho_i \min\limits_{1 \leq j \leq M_i} \mathrm{Re}\,(d_j^{i}/\delta_j^{i}) > 0 \quad (i = 1, 2)$

(iii) The $H$-function of two variables occurring in (3.1) satisfies the conditions corresponding appropriately to those given in the book [6,p.83,eqns.(6.1.5)-(6.1.6)]. ... (3.2)

**Proof :** To prove (3.1) we first express the $H$-function of two varaibels in terms of its Mellin-Barnes contour integrals with the help of (1.8) and interchange the order of $\xi$, $\eta$ and $x_1$, $x_2$ integrals (which is permissible under the conditions stated with the integral (3.1)), to get

$$\frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi(\xi,\eta)\, \phi_1(\xi)\, \phi_2(\eta)\, z_1^{\xi} z_2^{\eta}\, d\xi\, d\eta$$

$$\int_0^\infty \int_0^\infty (\lambda_1 x_1 + \lambda_2' x_2)^{\sigma_1 + \rho_1 \xi - 1} (\mu_1 x_1 + \mu_2 x_2)^{\sigma_2 + \rho_2 \eta - 1}$$

$$\exp\left[-s_1(\lambda_1 x_1+\lambda_2 x_2)-s_2(\mu_1 x_1+\mu_2 x_2)\right] S_V^U\left[\lambda_1 x_1+\lambda_2 x_2\right]$$

$$S_n^{\alpha,\beta,0}\left[\mu_1 x_1+\mu_2 x_2\right] dx_1\, dx_2 \qquad \ldots (3.3)$$

Now evaluating the above $x_1, x_2$ double integral with the help of (2.1) and reinterpreting the result, thus obtained in terms of the $H$-function of two variables, we easily arrive at the desired result after a little simplification.

## 4. SPECIAL CASES

(i) If we take $\lambda_1=\mu_2=1$, $\lambda_2=\mu_1=0$ in (3.1) and reduce the polynomial $S_V^U[x_1]$ in terms of Gould and Hopper polynomials $g_V^U[x_1, h]$ [4,p.58,eq.(6.2)] for which [see 14, p.161, eq.(1.15)] and the generalized polynomials set $S_n^{\alpha,\beta,0}[x_2]$ in terms of Gould and Hopper polynomials $H_n^{(r)}[x_2, \alpha, \beta]$ [u, p.52] for which [see, 9, p.65], we arrive at the following double Laplace transform :

$$L\left\{x_1^{\sigma_1-1} x_2^{\sigma_2-1} g_V^U[x_1, h]\, H_n^{(r)}[x_2, \alpha, \beta]\, H[z_1 x_1^{\rho_1}, z_2 x_2^{\rho_2}] : s_1, s_2\right\}$$

$$= \sum_{K=0}^{[V/U]} \sum_{v=0}^{n} \sum_{u=0}^{v} \frac{(UK)!\begin{pmatrix} V \\ UK \end{pmatrix} h^K (-1)^n (-v)_u (-\alpha-ur)_n\, \beta^v}{K!\, u!\, v!\, s_1^{\sigma_1'}\, s_2^{\sigma_2-n+vr}}$$

$$H^{O,N:M_1,N_1+1:M_2,N_2+1}_{P,Q:P_1+1,Q_1:P_2+1,Q_2}\left[\begin{matrix} z_1 s_1^{-\rho_1} \\ z_2 s_2^{-\rho_2} \end{matrix}\,\middle|\,\begin{matrix} (a_j:\alpha_j', \alpha_j'')_{1,P} : (1-\sigma_1', \rho_1), (c_j', \gamma_j')_{1,P_1}; \\ (b_j:\beta_j', \beta_j'')_{1,Q} : (d_j', \delta_j')_{1,Q_1}; \end{matrix}\right.$$

$$\left.\begin{matrix} (1-\sigma_2+n-vr, \rho_2), (c_j'', \gamma_j'')_{1,P_2} \\ (d_j'', \delta_j'')_{1,Q_2} \end{matrix}\right] \qquad \ldots (4.1)$$

where $\qquad \sigma_1' = \sigma_1 + (V-UK) \qquad \ldots (4.2)$

and the conditions easily obtainable from those stated with (3.1) are satisfied.

(ii) If we take $\lambda_1=\mu_2=1$, $\lambda_2=\mu_1=0$ in (3.1) and reduce the $H$-function of two variables in terms of product of a Whittaker and a modified Bessel's function [13, p.90, eq. (6.4.15), p.18, eqs. (2.6.7), (2.6.6)], the polynomials $S_V^U[x_1]$ in terms of Hermite polynomials $H_n(1/\sqrt{x_1})$ with the help of [14,p.158,eq.(1.4)] and the polynomial set $S_n^{\alpha,\beta,0}[x_2]$ in terms of Laguerre polynomials $L_n^{(\alpha)}[x_2]$ by taking $\beta=r=1$ in [9, p.65, eq. (2.1.15)], we arrive at a result given by Gupta and Agrawal [6, p.303, eq. (3.2)].

(iii) If in (3.1), we reduce the polynomials $S_V^U\,[\lambda_1 x_1 + \lambda_2 x_2]$ to Laguerre polynomials $L_V^{(p)}\,[\lambda_1 x_1 + \lambda_2 x_2]$ using [14, p.159, eq.11.8] and the generalized polynomial set $s_n^{\alpha,\,\beta,\,0}\,[\mu_1 x_1 + \mu_2 x_2]$ to polynomials $L_n^{(\alpha)}\,[\mu_1 x_1 + \mu_2 x_2, r, \beta]$ introduced by Singh and Srivastava [11] by using [9, p.65, eq. (2.1.15)], we arrive at the following integral

$$\int_0^\infty \int_0^\infty (\lambda_1 x_1 + \lambda_2 x_2)^{\sigma_1 - 1} (\mu_1 x_1 + \mu_2 x_2)^{\sigma_2 - 1}$$

$$\exp\,[-\, s_1(\lambda_1 x_1 + \lambda_2 x_2) - s_2(\mu_1 x_1 + \mu_2 x_2)]$$

$$L_V^{(p)}\,[\lambda_1 x_1 + \lambda_2 x_2]\, L_n^{(\alpha)}\,[\mu_1 x_1 + \mu_2 x_2, r, \beta]$$

$$\cdot H \begin{bmatrix} z_1\,(\lambda_1 x_1 + \lambda_2 x_2)^{\rho_1} \\ z_2\,(\mu_1 x_1 + \mu_2 x_2)^{\rho_2} \end{bmatrix} dx_1\, dx_2$$

$$= \frac{1}{R} \sum_{K=0}^{V} \sum_{v=0}^{n} \sum_{u=0}^{v} \frac{(-V)_k\,(-v)_u\,\beta^v\,(-\alpha - n - ru)_n}{K!\,u!\,v!\,(1+p)_K} \binom{V+p}{V}$$

$$H^{O,N:M_1,N_1+1:M_2,N_2+1}_{P,Q:P_1+1,Q_1\;:P_2+1,Q_2} \left[ \begin{matrix} z_1 s_1^{-\rho_1} \\ z_2 s_2^{-\rho_2} \end{matrix} \middle| \begin{matrix} (a_j : \alpha_j', \alpha_j'')_{1,P} : (1-\sigma_1-K, \rho_1), \\ (b_j : \beta_j', \beta_j'')_{1,Q} : (d_j', \delta_j')_{1,Q_1} \end{matrix} \right.$$

$$\left. \begin{matrix} (c_j', \gamma_j')_{1,P_1} \,; (1-\sigma_2 - vr, \rho_2), (c_j'', \gamma_j'')_{1,P_2} \\ ; (d_j'', \delta_j'')_{1,Q_2} \end{matrix} \right] \qquad \ldots (4.3)$$

where $R$ is defined as in (2.3) and the conditions easily derivable from (3.1) are satisfied.

Further, if we take $\sigma_1 = \sigma_2 = 1 = s_1 = s_2$ in (4.3), reduce the polynomials $L_n^{(\alpha)}\,[\mu_1 x_1 + \mu_2 x_2, r, \beta]$ to Laguerre polynomials $L_n^{(\alpha)}\,[\mu_1 x_1 + \mu_2 x_2]$ by taking $\beta = r = 1$ and the $H$-function of two variables into unity [By first reducing the $H$-function of two variables into product of two Fox's $H$-functions and then each of Fox's $H$-functions into exponential function with the help of results [13,p.90, eq.(6.4.15) and p.18, eq.(2.6.2)]. Next taking the limit as $z_1, z_2 \to 0$ and suitably specializing the parameters], we arrive at a known result given by Dhawan [2, p.417, eq. (2.2)] after a little simplification

A number of other results involving products of several simpler functions and simple polynomials, obtained as special cases of $H$-function of two variables and the general polynomials $S_U^V\,[x]$ and the polynomial set $s_n^{\alpha,\,\beta,\,0}\,[x]$ can be obtained as special cases of our main result (3.1) but we do not record them here on account of lack of space.

## REFERENCES

[1] S.K. Chatterjea, Quelques functions génératrices des polynômes d'Hermite, du poiunt de vue de l'algebre de Lie, *C.R. Acad. Sci. Paris Ser. A-B* **268** (1969), A600-A602.

[2] G.K. Dhawan, On integrals involving classical polynomials, *Proc. Camb. Phil. Soc.* **64** (1968), 417.

[3] S.S. Dhillon, *A study of Generalization of Special Functions of Mathematical Physics and Applications,* Ph.D. Thesis, Bundelkhand Univ., India, 1989.

[4] H.W. Gould and A.T. Hopper, Operational formulas connected with two generalizations of Hermite polynomials, *Duke Math. J.* 29 (1962), 51-63.

[5] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, Inc. New York, 1980.

[6] K.C. Gupta and Pawan Agrawal, A study of unified double and multivariable integrals and the Laplace transforms associated with them, *Bull. Cal. Math. Soc.* **87** (1995), 299-306.

[7] H.L. Krall and O. Frink, A new class of orthogonal polynomials : The Bessel polynomials, *Trans. Amer. Math. Soc.* **65** (1949), 100- 115.

[8] P.K. Mittal and K.C. Gupta, An integral involving generalized function of two variables, *Proc. Indian Acad. Sci.* **78**(A) (1972), 117-123.

[9] S.K. Raizada, *A Study of Unified Representation of Special Functions of Mathematical Physics and Their Use in Statistical and Boundary Value Problems*, Ph.D. Thesis, Bundelkhand Univ., India, 1991.

[10] A. Singh, *A Study of Special Functions of Mathematical Physics and Their Applications in Combinateral Analysis*, Ph.D. Thesis, Bundelkhand Univ., India, 1981.

[11] R.P. Singh and K.N. Srivastava, A note on generalization of Laguerre and Humbert polynomials, *Ricerco (Napoli)* (2) **14** (1963), 11-21; Errata, ibid. **15** (1964), 63.

[12] H.M. Srivastava, A contour integral involving Fox's $H$-function, *Indian J. Math.* **14** (1972), 1-6.

[13] H.M. Srivastava, K.C. Gupta and S.P. Goyal, *The H-Functions of One and Two Variables with Applications*, South Asian Publishers, New Delhi and Madras, 1982.

[14] H.M. Srivastava and N.P. Singh, The integration of certain products of multivariable $H$-function with a general class of polynomials, *Rend. Circ. Mat. Palermo* **32**(2) (1983), 157-187.

[15] D.V. Widder, *Advanced Calculus*, Dover Publications, Inc. New York, 1968.

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# SOME PROBABILITY DISTRIBUTIONS AND EXPECTATIONS ASSOCIATED WITH MULTIAVRIATE BETA AND GAMMA DISTRIBUTIONS INVOLVING MULTIPLE HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

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## ABSTRACT

In the present paper, we discuss some multivariate beta and gamma distributions and make their applications to derive their expectations in terms of multiple hypergeometric functions of several variables.

**1. Introduction.** Exton [6, p.222] studied many special multivariate distributions having expectations in terms of Lauricella's multiple hypergeometric functions [8]. Motivated by this work, here in the present paper, we discuss some multivariate beta and gamma distributions and make their applications to derive the expectations of different functions in terms of multiple hypergeometric functions ${}^{(k)}_{(1)}E^{(n)}_{C}$ of Chandel [1], ${}^{(k)}_{(1)}E^{(n)}_{D}$ , ${}^{(k)}_{(2)}E^{(n)}_{D}$ of Exton [5,6], ${}^{(k)}F^{(n)}_{AD}$ , ${}^{(k)}F^{(n)}_{BD}$ of Chandel and Gupta [3] and ${}^{(k)}F^{(n)}_{CD}$ of Karlsson [7]. Finally we also discuss their special cases.

**2. Expectations of different functions related to multivariate beta distributions**. The expectation for the function $f(x_1, \ldots, x_n)$ having multivariate density function $f(x_1, \ldots, x_n)$ is defined as

$$(2.1) < g\,(x_1\,, \ldots, x_n) > = \int_{-\infty}^{\infty} \ldots (n) \ldots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n) g\,(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

We consider the density function

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$$(2.2)\; f_1(x_1, \dots, x_n) = x_1^{b_1-1} \,..\, x_n^{\,b_n-1}\,(1-x_1-\dots-x_n)^{c-b_1-\dots-b_k-1}$$

$$\cdot\,(1-x_{k+1}-\dots-x_n)^{c'-b_{k+1}-\dots-b_n-1}$$

$$\cdot\,\frac{\Gamma(c)\Gamma(c')}{\Gamma(b_1)\dots\Gamma(b_n)\Gamma(c-b_1-\dots-b_k)\Gamma(c'-b_{k+1}-\dots-b_n)},$$

provided that $0 \le x_1,\dots,0 \le x_n$ , $x_1+\dots+x_k \le 1$ , $x_{k+1}+\dots+x_n \le 1$ and real parts of $c,c'$ , $c-b_1-\dots-b_k$ and $c'-b_{k+1}-\dots-b_n$ are positive and $f_1 = 0$ elsewhere.

Another density function is

$$(2.3)\; f_2(x_1, \dots, x_n) = x_1^{b_1-1} \dots x_n^{\,b_n-1}\,(1-x_1-\dots-x_k)^{c-b_1-\dots-b_k-1}$$

$$\cdot\,(1-x_{k+1})^{c_{k+1}-b_{k+1}-1} \dots (1-x_n)^{c_n-b_n-1}$$

$$\cdot\,\frac{\Gamma(c)\,\Gamma(a_{k+1})\dots\Gamma(a_n)}{\Gamma(b_1)\dots.\Gamma(b_n)\Gamma(c-b_1-\dots-b_k)\Gamma(c_{k+1}-b_{k+1})\dots\Gamma(c_n-b_n)}$$

where $x_1+\dots+x_k \le 1$ , $0 \le x_r \le 1$ , $r=1,\dots,n$

$\mathrm{Re}(c) > \mathrm{Re}(b_1+\dots+b_k)$ , $\mathrm{Re}(b_j) > 0$, j = 1, ...., k,

$\mathrm{Re}(c_i) > \mathrm{Re}(b_i) > 0$, $i=k+1,\dots,n$ and $f_2$ is zero elsewhere.

Corresponding to density function $f_1$ defined by (2.2) consider the function

$$(2.4)\qquad g_1(x_1, \dots, x_n) = (1-x_1\,\alpha_1-\dots-x_n\,\alpha_n)^{-a}.$$

Now putting the values of $f_1$ and $g_1$ from (2.2) and (2.4) respectively in (2.1) and making an appeal to the result due to Exton [6, p.93. (3.4.2.4)] we obtain the expectation of $g_1$ corresponding to function $f_1$ :

$$(2.5)\quad < E_{1,1} > {}^{(k)}_{(1)}E_D^{(n)}\,(a,b_1,\dots,b_n\,;c,c'\,;\alpha_1,\dots,\alpha_n),$$

where ${}^{(k)}_{(1)}E_D^{(n)}$ is multiple hypergeometric function related to Lauricella's $F_D^{(n)}$ introduced by Exton [5].

Further putting the values of $f_2$ and $g_1$ from (2.3) and (2.4) respectively in (2.1) and making an appeal to the result due to Karlsson [7.(3.2)], we derive following expectation value of $g_1$ corresponding to the function $f_2$

(2.6) $< E_{1,2} > {}^{(k)}F^{(n)}_{AD}\,(a,b_1,\ldots,b_n\,;c,c_{k+1},\ldots,c_n\,;\alpha_1,\ldots,\alpha_n)$

where ${}^{(k)}F^{(n)}_{AD}$ is one of intermediate Lauricella's multiple hypergeometric function introduced by Chandel and Gupta [3].

Now we consider the density function

(2.7) $$f_3\,(x_1,\ldots,x_n) = x_1^{b_1-1}\ldots x_n^{b_n-1}\,(1-x_1-\ldots-x_n)^{c-b_1-\ldots-b_n-1} \cdot \frac{\Gamma(c)}{\Gamma(b_1)\ldots.\Gamma(b_n)\Gamma(c-b_1-\ldots-b_n)}$$

provided that $0 \le x_1,\ldots,0 \le x_n\,,\, x_1+\ldots+x_n \le 1$ and all the real parts of c and $c-b_1-\ldots-b_n$ are positive, also $f_3 = 0$ else where.

Consider

(2.8) $$g_2\,(x_1,\ldots,x_n) = (1-\alpha_1x_1-\ldots-\alpha_kx_k)^{-a}\,(1-\alpha_{k+1}x_{k+1}-\ldots-\alpha_nx_n)^{-a'}.$$

Then putting these values of $f_3$ and $g_2$ from (2.7) and (2.8) respectively in (2.1) and making an appeal to the result due to Exton [6, p.93. (3.4.2.5)], we derive the following expectation of $g_2$ corresponding to function $f_3$ :

(2.9) $$< E_{2,3} > = {}^{(k)}_{(2)}E^{(n)}_{D}\,(a,a',b_1,\ldots,b_n\,;c;\,\alpha_1,\ldots,\alpha_n)$$

where ${}^{(k)}_{(2)}E^{(n)}_{D}$ is another multiple hypergeometric function related to Lauricella's $F^{(n)}_D$ introduced by Exton [5].

Further consider the function

(2.10) $$g_3\,(x_1,\ldots,x_n) = (1-\alpha_1x_1-\ldots-\alpha_kx_k)^{-a}\,(1-\alpha_{k+1}x_{k+1})^{-a_{k+1}}\ldots(1-\alpha_nx_n)^{-a_n}.$$

Now putting the value of $f_3$ and $g_3$ from (2.7) and (2.10) respectively in (2.1) and making an appeal to the result due to Karlsson [7,(3.1)], we get the expectation value of $g_3$ corresponding to the function $f_3$

(2.11) $$< E_{3,3} > = {}^{(k)}F^{(n)}_{BD}\,[a,a_{k+1},\ldots,a_n\,,b_1,\ldots,b_n\,;c\,;\alpha_1,\ldots,\alpha_n\,].$$

where ${}^{(k)}F^{(n)}_{BD}$ is one of intermediate Lauricella's multiple hypergeometric functions due to Chandel and Gupta [3].

**3. Expectations of different multiple hypergeometric functions related to multivariate gamma distributions.** In this section, we discuss some density functions associated with gamma distributions and derive some expectations involving multiple hypergeometric functions.

Consider the density function

$$(3.1)\quad f(x,y,z)=\frac{1}{\Gamma(a)\Gamma(a')\Gamma(b)}\,e^{-x-y-z}\,x^{a-1}\,y^{a'-1}\,z^{b-1},$$

provided that $0 \le x,y,z < \infty$, $\mathrm{Re}(a) > 0$, $\mathrm{Re}(a')>0$,$\mathrm{Re}(b)>0$ and $f(x,y,z)=0$ elsewhere.

Consider another function

$$(3.2)\quad g(x,y,z) = {}_0F_1(-;c_1;x_1\,xz)\ldots{}_0F_1(-;c_k;x_k\,xz)$$
$$ {}_0F_1(-;c_{k+1};x_{k+1}\,yz)\ldots{}_0F_1(-;c_n;x_n\,yz).$$

Now making an appeal to the result due to Exton [6, p.96 (3.4.4.6)], we derive the following expectations value of g(x, y, z) corresponding to the function f(x, y, z) :

$$(3.3)\quad < g(x,y,z) > = {}^{(k)}_{(1)}E^{(n)}_{C}\,(a,a',b;c_1,\ldots,c_n;x_1,\ldots,x_n),$$

where ${}^{(k)}_{(1)}E^{(n)}_{C}$ is multiple hypergeometric function related to Lauricella's $F^{(n)}_{C}$ introduced by Chandel [1].

Also consider the density function

$$(3.4)\; f_4(y,x_1,\ldots,x_n)=\frac{1}{\Gamma(c)\Gamma(b_1)\ldots\Gamma(b_n)}\,e^{-y-x_1-\ldots-x_n}\,y^{a-1}\,x_1^{b_1-1}\ldots x_n^{b_n-1},$$

provided that $0 < y, x_1,\ldots,x_n < \infty$, $\mathrm{Re}(a)$, $\mathrm{Re}(b_1),\ldots,\mathrm{Re}(b_n) > 0$ and $f_4 = 0$ elsewhere.

Further consider

$$(3.5)\quad f_4(y,x_1,\ldots,x_n) = {}_0F_1[-,c;\alpha_1 xy+\ldots+\alpha_k x_k\,y]$$
$$ {}_0F_1[-,c';\alpha_{k+1}x_{k+1}y+\ldots+\alpha_n x_n\,y].$$

Now putting the values of $f_4$ and $g_4$ from (3.4) and (3.5) respectively in (2.1) and making an appeal to the result due to Chandel and Gupta [2, (2.2)], we obtain the following expectation of $g_4$ corresponding to the function $f_4$ :

$$(3.6) < E_{4,4} > {}^{(k)}_{(1)}E^{(n)}_{D}\,(a,b_1,\ldots,b_n;c,c';\alpha_1,\ldots,\alpha_n).$$

Now we consider the density function

$$(3.7)\quad f_5\,(z,y,x_1\,,\,\dots\,,x_n) = \frac{1}{\Gamma(a)\Gamma(a')\Gamma(b_1)\dots\Gamma(b_n)}\,e^{-z-y-x_1-\,\dots\,-x_n}$$

$$z^{a-1}\,y^{a'-1}\,x_1^{b_1-1}\,\dots\,x_n^{b_n-1},$$

where $0 < z,y,x_1\,,\dots,\,x_n < \infty$ and all Re $(a)$ Re $(a')$ , Re$(b_1)$,...,Re$(b_n) > 0$ also $f_5 = 0$ elsewhere.

Consider the density function

$$(3.8)\; g_5\,(z,y,x_1\,,\,\dots\,,x_n) = {}_0F_1[-\,,c\,;(\,\alpha_1x_1 + \dots + \alpha_kx_k)\,z$$

$$+\,(\,\alpha_{k+1}x_{k+1} + \dots + \alpha_nx_n)y].$$

Now putting the values of $f_5$ and $g_5$ from (3.7) and (3.8) respectively in (2.1) and then making an appeal to the result due to Chandel and Gupta [2, (2.3)] we get the following expectation of $g_5$ corresponding to the function $f_5$ :

$$(3.9) < E_{5,5} > {}^{(k)}_{(2)}E_D^{(n)}\,(a,a',\,b_1\,,\dots,\,b_n\,;c,\,;\alpha_1\,,\,\dots\,,\,\alpha_n).$$

Further consider the density function

$$(3.10)\; f_6\,(x,x_{k+1}\,,\,\dots\,,x_n\,,y_1\,,\dots,y_n\,) = \frac{1}{\Gamma(a)\Gamma(a_{k+1})\dots\Gamma(a_n)\Gamma(b_1)\dots\Gamma(b_n)}$$

$$x^{a-1}\,x_{k+1}^{a_{k+1}-1}\,\dots\,x_n^{a_n-1}\,y_1^{b_1-1}\,\dots\,y_n^{b_n-1}\,e^{-(x+x_{k+1}+\dots+x_n+y_1+\dots+y_n)}\,,$$

where $0 \le x,x_{k+1},\dots,x_n\,,y_1,\dots,y_n < \infty$ and Re$(a)$ , Re $(a_{k+1})$, ..., Re$(a_n)$, Re $(b_1)$, ..., Re$(b_n) > 0$, also $f_6 = 0$ elsewhere.

Consider another density function

$$(3.11)\; g_6\,(x,x_{k+1}\,,\,\dots\,,x_n\,,y_1\,,\dots,y_n)$$

$$= {}_0F_1[-\,,c\,;\alpha_1x_1\,x + \dots + \alpha_kx_k\,x + \alpha_{k+1}x_{k+1}y_{k+1} + \dots + \alpha_nx_ny_n].$$

Now putting the values of $f_6$ and $g_6$ from (3.10) and (3.11) respectively in (2.1) and making an appeal to the result due to Chandel and Gupta [3,(5.5)], we obtain the following expectation of $g_6$ having density function $f_6$ :

$$(3.12) < E_{6,6} > {}^{(k)}F_{BD}^{(n)}\,[a,a_{k+1},\dots,\,a_n\,,\,b_1\,,\dots,\,b_n\,;c\,;\alpha_1\,,\,\dots\,,\,\alpha_n].$$

Further consider density function

$$(3.13)\ f_7(z,y,x_1\,,\,\dots\,,x_k) = \frac{1}{\Gamma(a)\Gamma(b)\Gamma(b_1)\dots\Gamma(b_k)}\, e^{-z-y-x_1-\dots-x_k}$$

$$.\, z^{a-1}\, y^{b-1}\, x_1^{b_1-1} \dots x_k^{b_k-1},$$

provided that $0 \le z,y,x_1\,,\dots,\,x_k < \infty$ and $\mathrm{Re}(a)$, $\mathrm{Re}(b)$, $\mathrm{Re}(b_1)$, ..., $\mathrm{Re}(b_k) > 0$, also $f_7 = 0$ elsewhere and

$$(3.14)\ g_7\,(z,y,x_1\,,\,\dots\,,x_k) = {}_0F_1(-\,,c\,;\alpha_1 x_1 z + \dots + \alpha_k x_k z)$$

$${}_0F_1\,(-\,;c_{k+1}\,;zy\alpha_{k+1}) \dots {}_0F_1(-\,;c_n\,;zy\alpha_n).$$

Now putting the values of $f_7$ and $g_7$ from (3.13) and (3.14) respectively in (2.1) and applying the result due to Chandel and Vishwakarma [4, p.178(3.9)], we derive corresponding expectation of $g_7$ :

$$(3.15) < E_{7,7} > {}^{(k)}F_{CD}^{(n)}\,(a,b\,,\,b_1\,,\dots,\,b_k\,;c,c_{k+1}\,,\dots,\,c_n\,;\ \alpha_1\,,\,\dots\,,\,\alpha_n)$$

where ${}^{(k)}F_{CD}^{(n)}$ is fourth possible intermediate Lauricella's multiple hypergeometric function due to Karlsson [8].

## 4. Special Cases.

**(4.1)** For $k = 0$, from (2.5) and (2.9), we deduce separately the result due to Exton [6, (7.2.1.5)].

**(4.2)** For $k = 0$, from (2.6), we derive the result due to Exton [6, (7.2.3.1)],

while

**(4.3)** for $k = n$, from (2.6), we obtain the result due to Exton [6,(7.2.1.5)]

**(4.4)** for $k = 0$, (2.11) gives the result due to Exton [6, (7.2.1.6)],

while

**(4.5)** for $k = n$, (2.11) gives the result due to Exton [6, (7.2.1.5)].

Further

**(4.6)** for $k = 0$, from (3.6) and (3.9), we deduce the result due to Exton [6, (7.2.3.2)].

Also

**(4.7)** for $k = n$, from (3.12) and (3.15), we derive the result due to Exton [6, (7.2.3.2)].

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## REFERRENCES

[1] R.C.S. Chandel, On some multiple hypergeometric functions related to Lauricella functions, *Jñānābha Sect. A*, **3**(1973), 119 - 136.

[2] R.C.S. Chandel and A.K.Gupta, Recurrence relations of hypergeometric functions of several variables, *Pure Appl. Math. Sci.*, **21**, 1-2 (1985), 65 - 70

[3] R.C.S. Chandel and A.K. Gupta, Multiple hypergeometric functions related to Lauricella's functions, *Jñānābha,* **16** (1986), 195 - 209.

[4] R.C.S. Chandel and P.K. Vishwakarma, Karlsson's multiple hypergeometric functions and its confluent forms, *Jñānābha,* **19** (1989), 173 - 185.

[5] H. Exton, On two multiple hypergeometric functions related to Lauricella's $F_D^{(n)}$ *Jñānābha Sect. A*, **2** (1972), 59 - 73.

[6] H. Exton, *Multiple Hypergeometric Functions and Applications*, John Wiley and Sons Inc., New York, 1976.

[7] Per W. Karlsson, On intermediate Lauricella's Functions, *Jñānābha,* **16** (1986) 211-222.

[8] G. Lauricella, Sulle funzioni ipergeometriche a piu variabili, *Rend Circ. Mat. Palermo,* **7** (1893), 111-158.

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# ON A UNIFIED FINITE INTEGRAL

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In the present paper, we first evaluate a unified finite integral involving the product of a general class of polynomials $S_V^U[x]$, the generalized polynomial set $S_n^{\alpha,\beta,\tau}[x]$ and the multivariable $H$-function. Being of very general nature, due to the presence of these polynomials and function, our result provides interesting unification and extension of a large number of new and known results obtained by several authors and hitherto lying scattered in the litrature. For the sake of illustration, we have evaluated here three new integrals as special cases of our main integral. Further, a reference to 12 integrals which also follow as simple special cases of our results has also been given.

## 1. INTRODUCTION

(*i*) Srivastava [3, eq. (1), p. 1] has introduced the following general class of polynomials :

$$S_V^U[x] \sum_{K=0}^{[V/U]} \frac{(-V)_{UK} A(V,K)\, x^K}{K!}, \quad V = 0, 1, 2, \ldots \qquad \ldots (1.1)$$

where $U$ is an arbitrary positive integer and the coefficients $A(V, K)(V, K \geq 0)$ are arbitrary constants real or complex. On suitably specializing the coefficients $A(V, K)$, $S_V^U[x]$ yields a number of known polynomials as its special cases. These include, among others, Laguerre polynomials, Hermite polynomials, Gould-Hopper polynomials, Brafman polynomials and several others [5, pp. 158-161].

(*ii*) The generalized polynomial set is defined by the following Rodrigues type formula [12, eq. (2.1.8), p.64].

$$S_n^{\alpha,\beta,\tau}[x : r, s, q, A, B, m, k, l]$$

$$= (Ax+B)^{-\alpha}\,(1-\tau x)^{-\beta/\tau}\,T_{k,\,l}^{m+n}\,[(Ax+B)^{\alpha+qn}\,(1-\tau x^{r})^{\beta/\tau+sn}] \qquad \ldots (1.2)$$

with the differential operator $T_{k,\,l}$ being defined as

$$T_{k,\,l} \equiv x^{l}\left(k + x\frac{d}{dx}\right) \qquad \ldots (1.3)$$

The explicit form of this generalized polynomial set [12, eq. (2.3.4), p. 71] is

$$S_n^{\alpha,\,\beta,\,\tau}\,[x : r, s, q, A, B, m, k, l]$$

$$= B^{qn}\,x^{l(m+n)}\,(1-\tau x^{r})^{sn}\,l^{m+n}\sum_{v=0}^{m+n}\;\sum_{e=0}^{v}\;\sum_{\delta=0}^{m+n}\;\sum_{p=0}^{\delta}\frac{(-1)^{\delta}\,(-\delta)_p}{p!\,\delta!} \qquad \ldots (1.4)$$

$$\frac{(\alpha)_\delta(-\nu)_e(-\alpha-qn)_p}{e!\,\nu!(1+\alpha-\delta)_p}\cdot(-\beta/\tau-\lambda_n)_\nu\left(\frac{p+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^{r}}{1-\tau x^{r}}\right)^{\nu}\left(\frac{Ax}{B}\right)^{\delta}$$

It may be pointed out here that the polynomial set defined by (1.2) is very general in nature and it unifies and extends a number of classical polynomials introduced and studied by various research workers such as Chatterjea [11], Dhillon [14], Gould and Hopper [7], Krall and Frink [2], Singh [1], Singh and Srivastava [10] etc. Some of the special cases of (1.3) are given by Raizada in a tabular form [12, p. 65].

(*iii*) The multivariable $H$-function introduced and studied by Srivastava and Panda [6], occurring in this paper will be defined and represented as follows [4, pp. 251-252, eqns. (C.1)-(C.3)].

$$H_{P,\,Q\,:\,P_1,\,Q_1\,;\ldots;\,P_t,\,Q_t}^{O,\,N\,:\,M_1,\,N_1\,;\ldots;\,M_t,\,N_t}\left[\begin{matrix} z_1 \\ \cdot \\ \vdots \\ z_t \end{matrix}\;\middle|\;\begin{matrix} (a_j : \alpha_j^{(1)},\ldots,\alpha_j^{(t)})_{1,\,P} : \\ (b_j : \beta_j^{(1)},\ldots,\beta_j^{(t)})_{1,\,Q} : \end{matrix}\right.$$

$$\left.\begin{matrix} (c_j^{(1)},\,\gamma_j^{(1)}{}_{1,\,P_1}\,;\ldots;\,(c_j^{(t)},\,\gamma_j^{(t)})_{1,\,P_t} \\ (d_j^{(1)},\,\delta_j^{(1)})_{1,\,Q_1}\,;\ldots;\,(d_j^{(t)},\,\delta_j^{(t)})_{1,\,Q_t} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^{t}}\int_{L_1}\cdots\int_{L_t}\phi_1(\xi_1)\ldots\phi_t(\xi_t)\,\psi(\xi_1,\ldots,\xi_t)$$

$$z_1^{\xi_1}\ldots z_t^{\xi_t}\,d\xi_1\ldots d\xi_t \qquad \ldots (1.5)$$

where $\omega = \sqrt{-1}$

$$\phi_i(\xi_i) = \frac{\prod_{j=1}^{M_i} \Gamma(d_j^{(i)} - \delta_j^{(i)}\xi_i) \prod_{j=1}^{N_i} \Gamma(1 - c_j^{(i)} + \gamma_j^{(i)}\xi_i)}{\prod_{j=M_i+1}^{Q_i} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)}\xi_i) \prod_{j=N_i+1}^{P_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)}\xi_i)}$$

$$(i = 1, \ldots, t) \quad \ldots (1.6)$$

and

$$\psi(\xi_1, \ldots, \xi_t) = \frac{\prod_{j=1}^{N} \Gamma\left(1 - a_j + \sum_{i=1}^{t} \alpha_j^{(i)}\xi_i\right)}{\prod_{j=N+1}^{P} \Gamma\left(a_j - \sum_{i=1}^{t} \alpha_j^{(i)}\xi_i\right) \prod_{j=1}^{Q} \Gamma\left(1 - b_j + \sum_{i=1}^{t} \beta_j^{(i)}\xi_i\right)}$$

$$\ldots (1.7)$$

All the Greek letters occuring on the left hand side of (1.5) are assumed to be positive real numbers for standardization purposes; the definition of the multivariable $H$-function will, however, be meaningful even if some of these quantities are zero. For the convergence and existence conditions of the multivariable $H$-function we refer to the book by Srivastava et al. [4, eqns. (C.4)-(C.8), pp. 252-253]. Throughout the paper it is assumed that this function satisfies the above cited conditions.

## 2. MAIN INTEGRAL

$$\int_0^a x^{\rho-1} (a-x)^{\sigma-1} (c+bx)^{-\eta} S_n^{\alpha,\beta,\tau} [y x^{\mu} (a-x)^{\nu} (c+bx)^{-\lambda}]$$

$$S_V^U [z x^{\zeta} (a-x)^{\xi} (c+bx)^{-\gamma}] \, H [z_1 x^{P_1} (a-x)^{q_1} (c+bx)^{-r_1} \ldots$$

$$z_t x^{P_t} (a-x)^{q_t} (c+bx)^{-r_t}] \, dx$$

$$= \frac{a^{\rho+\sigma-1}}{c^{\eta}} \sum_{v=0}^{m+n} \sum_{e=0}^{v} \sum_{\delta=0}^{m+n} \sum_{p=0}^{\delta} \sum_{K=0}^{[V/U]} \theta(k, e, v, \delta, p)$$

$$H^{O,N+3:M_1,N_1;\ldots;M_t,N_t:1,1;1,0}_{P+3,Q+2:P_1,Q_1;\ldots;P_t,Q_t:1,1;0,1} \left[ z_1 \frac{a^{p_1+q_1}}{c^{r_1}}, \ldots, z_t \frac{a^{p_t+q_t}}{c^{r_t}}, \right.$$

$$\left. \frac{-\tau y^r a^{(\mu+\nu)r}}{c^{\lambda r}}, \frac{ab}{c} \,\middle|\, \begin{matrix} C : E ; (1-v+sn, 1) : - \\ D : F ; (0, 1) \qquad\qquad : (0, 1) \end{matrix} \right] \ldots (2.1)$$

where

$$\theta(K, e, v, \delta, p) = B^{qn}\, l^{m+n}\, \frac{a^{l(\mu+\nu)(m+n)}\, (-1)^{\delta}\, (-\delta)_p\, (\alpha)_\delta}{c^{\lambda l(m+n)}\, p!\, \delta!\, e!\, v!\, K!}$$

$$\frac{(-v)_e\, (-\alpha - qn)_p\, (-V)_{UK}\, A(V, K)\, z^{K}\, z^{K(\zeta+\xi)}}{(1-\alpha-\delta)_p\, \Gamma(v - sn)\, c^{\gamma K}}\, (-\beta/\tau - sn)_v$$

$$\left(\frac{p+k+re}{l}\right)_{m+n} + \left(\frac{-\tau a^{(\mu+\nu)}}{c^{\lambda r}}\right)^{v} \left(\frac{A a^{(\mu+\nu)}}{B c^{\lambda}}\right)^{\delta} \qquad \ldots (2.2)$$

$$C = (1-\rho-\mu R-\zeta K : p_1, \ldots, p_t, \mu r, 1), (1-\sigma-\nu R-\xi K : q_1, \ldots, q_t, \nu r, 0),$$

$$(1-\eta-\lambda R-\gamma K : r_1, \ldots, r_t, \lambda r, 1), (a_j : \alpha_j^{(1)}, \ldots, \alpha_j^{(t)}, 0, 0)_{1,P}$$

$$D = (b_j : \beta_j^{(1)}, \ldots, \beta_j^{(t)}, 0, 0)_{1,Q}, (1-\eta-\lambda R-\gamma K : r_1, \ldots, r_t, \lambda r, 0),$$

$$(1-\rho-\sigma-(\mu+\nu)R-(\zeta+\xi)K : p_1+q_1, \ldots, p_t+q_t, (\mu+\nu)r, 1)$$

$$E = (c_j^{(1)}, \gamma_j^{(1)})_{1,P_1} ; \ldots ; (c_j^{(t)}, \gamma_j^{(t)})_{1,P_t}$$

$$F = (d_j^{(1)}, \delta_j^{(1)})_{1,Q_1} ; \ldots ; (d_j^{(t)}, \delta_j^{(t)})_{1,Q_t} \qquad \ldots (2.3)$$

$$R = l(m+n) + rv + \delta \qquad \ldots (2.4)$$

and the following conditions are satisfied :

(*i*) $a, b, c$ are positive numbers such that $\left|\dfrac{ab}{c}\right| < 1$,

$\mathrm{Re}(\eta) > 0$, $\min\{\mu, \nu, \xi, \lambda, \zeta, \gamma, p_i, q_i, r_i\} \geq 0$, $i = 1, \ldots, t$

(not all zero simultaneously)

(*ii*) $$\mathrm{Re}(\rho) + \sum_{i=1}^{t} p_i \min_{1 \leq j \leq M_i} \left\{\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right\} > 0,$$

$$\mathrm{Re}(\sigma) + \sum_{i=1}^{t} q_i \min_{1 \leq j \leq M_i} \left\{\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right\} > 0,$$

$(i = 1, \ldots, t)$ ... (2.5)

**Proof .** To establish the integral (2.1), we first express the polynomials $S_V^U[x]$ and $S_n^{\alpha,\beta,\tau}[x]$ occurring in the left-hand side of (2.1) in their respective series forms with the help of equations (1.1) and (1.4) and then interchange the order of summations and integration (which is permissible under the conditions stated with

(2.1)). The left-hand side of the integral (2.1) (say $I$), then takes the following form :

$$I = B^{qn}\, l^{m+n} \sum_{v=0}^{m+n} \sum_{e=0}^{v} \sum_{\delta=0}^{m+n} \sum_{p=0}^{\delta} \sum_{K=0}^{[V/U]} \sum_{u=0}^{\infty} \frac{(-1)^{\delta}(-\delta)_p(-v)_e(\alpha)_{\delta}}{p!\,\delta!\,e!\,v!\,K!}$$

$$\frac{(-1)^{v}(-\alpha-qn)_p\, y^{R+ur}(-V)_{KU}(v-sn)_u(\tau)^{u+v}A(V,K)\,z^{K}}{u!\,(1-\alpha-\delta)_p}$$

$$(-\beta/\tau - sn)_v \left(\frac{p+k+re}{l}\right)_{m+n} \left(\frac{A}{B}\right)^{\delta} \int_0^a x^{\rho+\mu(R+ur)+\zeta K-1}$$

$$(a-x)^{\sigma+\nu(R+ur)+\xi K-1}\,(c+bx)^{-\eta-\lambda(R+ur)-\gamma K}$$

$$H\,[z_1 x^{p_1}(a-x)^{q_1}(c+bx)^{-r_1}, \ldots, z_t x^{p_t}(a-x)^{q_t}(c+bx)^{-r_t}]\,dx \qquad \ldots (2.6)$$

Evaluating the $x$-integral occurring in the above equation (2.6) with the help of [9, eq. (5.2.6), p. 244] and expressing the result thus obtained in terms of $H$-function of $(t+2)$ variables, we arrive at the desired result (2.1).

## 3. SPECIAL CASES

($i$) If we take $A = 1$, $B = q = s = m = k = 0$, $l = -1$ and let $\tau \to 0$ in the main integral (2.1), the polynomial set $S_n^{\alpha,\beta,\tau}[x]$ reduces into Gould and Hopper polymials $H_n^{(r)}[x, \alpha, \beta]$ [7] and we arrive at the following interesting integral after a little simplification :

$$\int_0^a x^{\rho-1}(a-x)^{\sigma-1}(c+bx)^{-\eta}\,H_n^{(r)}[yx^{\mu}(a-x)^{\nu}(c+bx)^{-\lambda}, \alpha, \beta]$$

$$S_V^U[zx^{\zeta}(a-x)^{\xi}(c+bx)^{-\gamma}]\,H\left[\frac{z_1 x^{P_1}(a-x)^{q_1}}{(c+bx)^{r_1}}, \ldots, \frac{z_t x^{p_t}(a-x)^{q_t}}{(c+bx)^{r_t}}\right]dx$$

$$= \frac{a^{\rho+\sigma-1}}{c^{\eta}} \sum_{v=0}^{n} \sum_{e=0}^{v} \sum_{K=0}^{[V/U]} \theta_1(K, e, v)$$

$$H^{O,N+3:M_1,N_1;\ldots;M_t,N_t;1,0}_{P+3,Q+2:P_1,Q_1;\ldots;P_t,Q_t;0,1}\left[\begin{matrix}\dfrac{z_1 a^{p_1+q_1}}{c^{r_1}}, \ldots, \dfrac{z_t a^{p_t+q_t}}{c^{r_t}}, \dfrac{ab}{c}\end{matrix}\,\middle|\, \begin{matrix} C_1 : E : - \\ D_1 : F : (0,1)\end{matrix}\right] \qquad \ldots (3.1)$$

where

$$\theta_1(K, e, v) = \frac{(-1)^n\,(-v)_e\,(-\alpha - re)_n\,\beta^v\,a^{(\mu+\nu)R_1 + K(\zeta+\xi)}\,(-V)_{UK}\,A(V, K)z^K}{e!\,v!\,K!\,c^{\lambda R_1 + \gamma K}}. \qquad \ldots (3.2)$$

$$C_1 = (1 - \rho - \mu R_1 - \zeta K : p_1, \ldots, p_t, 1), (1 - \sigma - \nu R_1 - \xi K : q_1, \ldots, q_t, 0),$$

$$(1 - \eta - \lambda R_1 - \gamma K : r_1, \ldots, r_t, 1), (a_j : \alpha_j^{(1)}, \ldots, \alpha_j^{(t)}, 0)_{1, P}$$

$$D_1 = (b_j : \beta_j^{(1)}, \ldots, \beta_j^{(t)}, 0)_{1, Q}, (1 - \eta - \lambda R_1 - \gamma K : r_1, \ldots, r_t, 0),$$

$$(1 - \rho - \sigma - (\mu + \nu)R_1 - (\zeta + \xi)K : p_1 + q_1, \ldots, p_t + q_t, 1) \quad \ldots (3.3)$$

$$R_1 = rv - \eta$$

provided that the conditions easily obtainable from those mentioned with the main integral (2.1) are satisfied.

(*ii*) Taking $\eta = \rho + \sigma$, $\lambda = \mu + \nu$, $\gamma = \zeta + \xi$, $r_1 = p_i + q_i$ $(i = 1, \ldots, t)$ and replacing $x$ by $ax/(1 + x)$ in (2.1), after making a little simplifications, we arrive at the following infinite form of the integral which itself is quite general in nature.

$$\int_0^\infty x^{\rho - 1}(1 + dx)^{-\rho - \sigma}\, S_n^{\alpha, \beta, \tau}[y_1 x^\mu (1 + dx)^{-\mu - \nu}]$$

$$S_V^U[z' x^\zeta (1 + dx)^{-\zeta - \xi}]\, H[z_1' x^{p_1}(1 + dx)^{-p_1 - q_1}, \ldots, z_t' x^{p_t}(1 + dx)^{-p_t - q_t}]\, dx$$

$$= d^{-\rho} \sum_{v=0}^{m+n} \sum_{e=0}^{v} \sum_{\delta=0}^{m+n} \sum_{p=0}^{\delta} \sum_{K=0}^{[V/U]} \theta_2(K, e, v, \delta, p)$$

$$H^{O, N+2 : M_1, N_1; \ldots; M_t, N_t; 1, 1}_{P+2, Q+1 : P_1, Q_1; \ldots; P_t, Q_t; 1, 1}\left[ z_1' d^{-p_1}, \ldots, z_t' d^{-p_t}, -\tau y_1 d^{-\mu r} \,\middle|\, \begin{matrix} C_2 : E : (1 - v + sn, 1) \\ D_2 : F : (0, 1) \end{matrix} \right] \quad \ldots (3.5)$$

where

$$d = \frac{c + ab}{c}, \; y_1' = y\left(\frac{a}{c}\right)^{\mu + \nu}, \; z' = z\left(\frac{a}{c}\right)^{\zeta + \xi},$$

$$z_i' = z_i\left(\frac{a}{c}\right)^{p_i + q_i} \text{ for } i = 1, \ldots, t \qquad \ldots (3.6)$$

$$\theta_2(K, e, v, \delta, p) = B^{qn}\, l^{m+n}\, \frac{(-1)^\delta\,(-\delta)_p\,(-v)_e\,(\alpha)_\delta\,(-\alpha - qn)_p}{p!\,\delta!\,e!\,v!\,K!\,\Gamma(v - sn)}$$

$$\frac{(-V)_{UK}\, A(V,K)\, (z')^{K}\, (-\beta/\tau - sn)_{v}\, (-\tau)^{V}}{(1-\alpha-\delta)_{p}} \left(\frac{a}{c}\right)^{(\mu+\nu)R} \left(\frac{A}{B}\right)^{\delta} . \left(\frac{p+k+re}{l}\right)_{m+n} \qquad \ldots (3.7)$$

$$C_2 = (1-\rho-\mu R-\zeta K : p_1, \ldots, p_t, \mu r), (1-\sigma-\nu R-\xi K : q_1, \ldots, q_t, \nu r) \; (a_j : \alpha_j^{(1)}, \ldots, \alpha_j^{(t)}, 0)_{1,p}$$

$$D_2 = (b_j : \beta_j^{(1)}, \ldots, \beta_j^{(t)}, 0)_{1,Q}, (1-\rho-\sigma-(\mu+\nu)R-(\zeta+\xi)K : p_1+q_1, \ldots, p_t+q_t, (\mu+\nu)r \qquad \ldots (3.8)$$

$$R = l(m+n) + rv + \delta \qquad \ldots (3.9)$$

provided that the conditions easily obtainable from those mentioned with the main integral (2.1) are satisfied.

Further, if we take $A=1, B=0=q$, $m=-n$, let $\tau \to 0$ and $V=0$, both the polynomials $S_n^{\alpha,\beta,\tau}[x]$ and $S_V^U[x]$ reduce into unity. Again, we put $t=3$ and reduce the $H$-function of 3-variables thus obtained into the product of $H$-function of one variable and $H$-function of two variables by taking $\alpha_j^{(1)} = \beta_j^{(1)} = 0$. Finally taking $\rho+\sigma=\eta$, $p_i+q_i=r_i$ $(i=1,2,3)$, $N_i=0$, $p_1=1$, $r_1=0$, $r_2=p_2$, $r_3=p_3$. We arrive at a known result [4, eq. (8.2.4), p. 136] after making some suitable adjustments and simplifications on the right-hand side.

(*iii*) If we take $\eta=\rho+\sigma$, $\lambda=\mu+\nu$, $\gamma=\zeta+\xi$, $r_i=p_i+q_i$ $(i=1,\ldots,t)$, $c=a$, $b=(f-g)/(h-f)$ and replace $x$ by $a\,(f-x)/(f-g)$ in the main integral (2.1), we arrive at the following form of the integral after a little simplification

$$\int_g^{\pounds} (f-x)^{\rho-1}(x-g)^{\sigma-1}(x-h)^{-\rho-\sigma} S_n^{\alpha,\beta,\tau}\left[\frac{y(f-x)^{\mu}(x-g)^{\nu}}{(x-h)^{\mu+\nu}}\right]$$

$$S_V^U\left[z\,\frac{(f-x)^{\zeta}(x-g)^{\xi}}{(x-h)^{\zeta+\xi}}\right] H\left[z_1\,\frac{(f-x)^{p_1}(x-g)^{q_1}}{(x-h)^{p_1+q_1}}, \ldots, z_t\,\frac{(f-x)^{p_t}x-g)^{q_t}}{(x-h)^{p_t+q_t}}\right] dx.$$

$$= \sum_{v=0}^{m+n} \sum_{e=0}^{v} \sum_{\delta=0}^{m+n} \sum_{p=0}^{\delta} \sum_{K=0}^{[V/U]} \theta_3(K, e, v, \delta, p)$$

$$H^{O,\,N+2\,:\,M_1,N_1;\ldots;M_t,N_t;\,1,1}_{P+2,\,Q+1\,:\,P_1,Q_1;\ldots;P_t,Q_t;\,1,1}$$

$$\left[z_1\left(\frac{f-g}{f-h}\right)^{p_1}\left(\frac{f-g}{g-h}\right)^{q_1},\ldots,z_t\left(\frac{f-g}{f-h}\right)^{p_t}\left(\frac{f-g}{g-h}\right)^{q_t},-\tau y^r\left(\frac{f-g}{f-h}\right)^{\mu r}\left(\frac{f-g}{g-h}\right)^{r}\right.$$

$$\left|\begin{matrix} C_3 : E : (1-\upsilon+sn,\,1) \\ D_3 : F : (0,\,1)\end{matrix}\right] \quad \ldots (3.10)$$

where

$$Y = y\left(\frac{f-h}{f-g}\right)^{\mu+\nu}, Z = z\left(\frac{f-h}{f-h}\right)^{\zeta+\xi}, Z_i = z_i\left(\frac{f-h}{f-g}\right)^{p_i+q_i}$$

$$(i = 1, \ldots, t) \quad \ldots (3.11)$$

$$\theta_3\,(K, e, \upsilon, \delta, p) = B^{qn}\, l^{m+n}\,\frac{(-1)^{\delta}(-\delta)_p\,(\alpha)_\delta\,(-\upsilon)_e\,(-\alpha-qn)_p}{p!\,\delta!\,e!\,\upsilon!\,K!\,(1-\alpha-\delta)_p}$$

$$\frac{(-\beta/\tau-sn)_\upsilon\,(-\tau)^\upsilon\,(-V)_{UK}\,A(V,K)\,Z^k}{\Gamma(\upsilon-sn)}\left(\frac{A}{B}\right)^{\delta}\left(\frac{f-g}{f-h}\right)^{\zeta}\left(\frac{f-g}{g-h}\right)^{\xi}$$

$$\left(\frac{p+k+re}{l}\right)_{m+n} \quad \ldots (3.12)$$

$C_3, D_3, R$ can be obtained from the equations (2.3) and (2.4) respectively. Provided that the conditions easily obtainable from those mentioned with the main integral (2.1) are satisfied.

Now, if we take $A = 1, B = 0 = q, m = -n$ and let $\tau \to 0$ in (3.10) the polynomial set $S_n^{\alpha,\beta,\tau}[x]$ reduces into unity. Further, change $t$ to $(t+1)$ and reduce the $H$-function of $(t+1)$ variables into the product of $H$-function of one variable $H$-function of $t$-variables by taking $\alpha_j^{(t+1)} = 0 = \beta_j^{(t+1)}$ therein, we arrive at an integral obtained by Chaurasia and Sharma [15, eq. (2.1), p. 42] after a little simplification.

(*iv*) If we take $A = 1, B = 0 = q, m = -n$ and let $\tau \to 0$ in the main integral (2.1), the polynomial set $S_n^{\alpha,\beta,\tau}[x]$ reduces into unity and we arrive at an integral essentially similar to the one obtained by Agrawal [13, eq. (4.5.1), p. 122].

Further, if we put $\eta = 0$, $\gamma = 0$, $r_i = 0$ $(i = 1, \ldots, t)$ change $t$ to $(t+1)$ and reduce the $H$-function of $(t+1)$ variables into the product of $H$-function of one variable and the $H$-function of $t$-variables by taking $\alpha_j^{(t+1)} = 0 = \beta_j^{(t+1)}$, we arrive at another known integral given by Chaurasia and Sharma [15, eq. (2.2), p. 43] after a little simplification. Further, if we take $t = 2$ and $V = 0$ reduce the polynomial $S_0^U[x]$ into unity, we easily arrive at another known integral [4, eq. (8.2.7), p. 138].

The results obtained earlier by Gupta et al. [8, eq. (3.1), p. 69], Srivastava and Singh [5, eq. (2.2), p. 166], Prasad and Singh [16, p.126] and Chaurasia and Sharma [15, eq. (2.3), p. 44 and eq. (2.4), p. 45] can also be obtained as simple special cases of the result referred in the above paragraphs. However, we omit the details here.

($v$) If in the main result (2.1) we reduce both the polynomials $S_V^U[x]$ and $S_n^{\alpha,\beta,\tau}[x]$ into unity by the substitutions indicated in special case ($ii$), we arrive at an integral same in essence obtained by Garg [9, eq. (5.2.6), p. 244]. Further, if we take $t=2$, $\eta=\rho+\sigma$, $r_i=p_i+q_i\ (i=1,2)$, c=a, b = $\frac{c-d}{l+d}$ and replace $x$ by $ax$, we get a known integral [4, eq. (8.4.4), p.144] after a little simplification.

Several other interesting results involving a large variety of polynomials, obtained as special cases of $S_V^U[x]$ and $S_n^{\alpha,\beta,\tau}[x]$, and various simpler functions, which are particular cases of the multivariable $H$-function, can also be obtained as special cases of our main result (2.1) but we do not record them here for lack of space.

## REFERENCES

[1] A. Singh, *A Study of Special Functions of Mathematical Physics and Their Applications in Combinateral Analysis,* Ph.D. Thesis, Bundelkhand Univ., Jhansi, India (1981).

[2] H.L. Krall and O. Frink, A new class of polynomials : The Bessel polynomials, Trans. Amer. Math. Soc., **65** (1949), 100-115.

[3] H.M. Srivastava, A countour integral involving Fox's $H$-function, *Indian J. Math.*, **14** (1972), 1-6.

[4] H.M. Srivastava, K.C. Gupta and S.P. Goyal, *The H-Functions of One and Two Variables with Applications*, South Asian Publishers, New Delhi and Madras, 1982.

[5] H.M. Srivastava and N.P. Singh, The integration of certain products of the multivariable H-function with a general class of polynomials, *Rend. Circ. Mat. Palermo* (2) **32** (1983), 157-187.

[6] H.M. Srivastava and R. Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials, *J. Reine Angew. Math.* **283/284** (1976), 265-274.

[7] H.W. Gould and A.T. Hopper, Operational formulas connected with two generalizations of Hermite polynomials, *Duke Math*. J. **29** (1962), 51-63.

[8] K.C. Gupta, S.P. Goyal and S. Verma, Unified integrals involving general class of polynomials and the mutlivariable *H-function, Indian J. Pure Appl. Math.* **20** (1989), 604-608.

[9] M. Garg, *A Study of Integral Operators, Fractional Calculus and H-Functions with Applications,* Ph.D. Thesis, Univ. of Raj., India (1983).

[10] R.P. Singh and K.N. Srivastava, A note on generalization of Laguerre and Humbert polynomials, *Ricerca (Napoli)*, (2) **14** (1963), 11-21, Errata, Ibid., 15 (1964), 63.

[11] S.K. Chatterjea, Quelques functions génératrices des polynômes d'Hermite, du point de vue de l'algèbre de lie, *C.R. Acad. Sci. Paris Ser.* A-B **268** (1969), A 600 - A 602.

[12] S.K. Raizada, *A Study of Unified Representation of Special Functions of Mathematical Physics and Their Use in Statistical and Boundary Value Problems*, Ph.D. Thesis, Bundelkhand Univ., India (1991).

[13] S.M. Agrawal, *A Study of Single and Mutlivariable H-Functions, Fractional Calculus and Integral Transforms with Applications,* Ph.D. Thesis, Univ. of Raj., India (1988).

[14] S.S. Dhillon, *A Study of Generalization of Special Functions of Mathematical Physics and Applications,* Ph.D. Thesis, Bundelkhand Univ., India (1989).

[15] V.B.L. Chaurasia and Rajendra Pal Sharma, Some integrals involving a general class of polynomials, Fox's *H*-function and the multivariable *H-function, Jñānābha*, **23** (1993), 40-49.

[16] Y.N. Prasad and S.N. Singh, Some expansion formulae for the *H*- function of two variables invo,lving Jacobi polynomials, *Bull. Math. Soc. Sci. Math.* R.S. Roumanie (N.S.), **21** (1977), 123-132.

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# QUASI-HADAMARD PRODUCT OF CERTAIN ANALYTIC FUNCTIONS

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## ABSTRACT

We consider the class $F_p^c(\lambda, \alpha)$ of certain analytic functions. It is shown that if $f \in F_p^{c_1}(\lambda_1, \alpha_1)$, $0 \le \lambda_1 < 1$, and $g \in F_p^{c_2}(\lambda_2, \alpha_2)$, $\lambda_2 \ge 1$, then the quasi-Hadamard product $f*g$ belongs to the class $F_p^{c_1+c_2}(\lambda_1, \alpha_1)$ or $F_p^{c_1+c_2}(\lambda_1, \alpha_2)$ according as $\alpha_1 \le \alpha_2$ or $\alpha_1 \ge \alpha_2$, respectively. This result fills a gap in a recent result of Subhas S. Bhoosnurmath and S.R. Swamy [Pure Appl. Math. Sci. **35** (1992), 71- 77].

## 1. INTRODUCTION

Let $T_p$ denote the class of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \; p \in N, \; a_{p+n} \ge 0,$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$.

If $f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ and $g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n}$ be any two functions in $T_p$, then their quasi-Hadamard product, denoted by $f*g$, is defined by (see [3])

$$(f*g)(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}$$

A function $f$ in $T_p$ belongs to the class $F_p^c(\lambda, \alpha)$ if and only if

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^c (p + n\lambda)\, a_{p+n} \le p - \alpha,$$

where $c \ge 0$, $\lambda \ge 0$ and $0 \le \alpha < p$.

It is easy to establish the following containment relations :

(*i*) $F_p^c(\lambda, \alpha_2) \subset F_p^c(\lambda, \alpha_1)$, if $\alpha_1 < \alpha_2$.

(*ii*) $F_p^c(\lambda_2, \alpha) \subset F_p^c(\lambda_1, \alpha)$, if $\lambda_1 < \lambda_2$.

(*iii*) $F_p^{c_2}(\lambda, \alpha) \subset F_p^{c_1}(\lambda, \alpha)$, if $c_1 < c_2$.

Bhoosnurmath and Swamy [1] introduced the above class and proved the following result :

**Theorem.** *If* $f \in F_p^{c_1}(\lambda_1, \alpha_1)$ *and* $g \in F_p^{c_2}(\lambda_2, \alpha_2)$, *then the quasi-Hadamard product f*g belongs to* $F_p^{c_0}(\lambda_0, \alpha_0)$, *where*

$$F_p^{c_0}(\lambda_0, \alpha_0) \equiv F_p^{c_1+c_2+1}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2+1}(\lambda_2, \alpha_2), \textit{ if } \lambda_1, \lambda_2 \geq 1$$

*and*

$$F_p^{c_0}(\lambda_0, \alpha_0) \equiv F_p^{c_1+C_2}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2}(\lambda_2, \alpha_2), \textit{ if } 0 \leq \lambda_1, \lambda_2 < 1.$$

The technique used by Bhoosnurmath and Swamy [1] is the same as employed by the author [2].

We note that their is a gap in the above result, *i.e.*, the case when any one of $\lambda_1$ and $\lambda_2$ is less than 1 and the other is greater than or equal to 1. It is the purpose of this paper to fill up this gap.

## 2. MAIN RESULT

**Theorem.** *If* $f \in F_p^{c_1}(\lambda_1, \alpha_1)$ *and* $g \in F_p^{c_2}(\lambda_2, \alpha_2)$, *where* $0 \leq \lambda_1 < 1$ *and* $\lambda_2 \geq 1$, *then the quasi-Hadamard product f*g belongs to* $F_p^{c_1+c_2+1}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2}(\lambda_2, \alpha_2)$.

**Proof.** $f \in F_p^{c_1}(\lambda_1, \alpha_1)$ implies $f \in F_p^{c_1}(0, 0)$ so that

$$\sum_{n=1} \left(\frac{n+p}{p}\right)^{c_1} a_{p+n} \leq 1.$$

Therefore

$$\left(\frac{n+p}{p}\right)^{c_1} a_{p+n} \leq 1, \quad \text{for all } n = 1, 2, 3, \ldots \qquad \ldots (2.1)$$

Further, $g \in F_p^{c_2}(\lambda_2, \alpha_2)$ implies that $g \in F_p^{c_2}(1, 0)$ so that

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^{c_2+1} b_{p+n} \leq 1.$$

Therefore

$$\left(\frac{n+p}{p}\right)^{c_2+1} b_{p+n} \leq 1, \text{ for all } n = 1, 2, 3, \ldots \qquad \ldots (2.2)$$

Using (2.2) we get

$$\begin{aligned}\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^{c_1+c_2+1} & (p+n\lambda_1)\, a_{p+n}\, b_{p+n} \\ & \leq \sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^{c_1} (p+n\lambda_1)\, a_{p+n} \\ & \leq p - \alpha_1, \quad \text{since } f \in F_p^{c_1}(\lambda_1, \alpha_1).\end{aligned}$$

This shows that $f{*}g \in F_p^{c_1+c_2+1}(\lambda_1, \alpha_1)$.

Similarly, by using (2.1) we get

$$\begin{aligned}\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^{c_1+c_2} & (p+n\lambda_2)\, a_{p+n}\, b_{p+n} \\ & \leq \sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^{c_2} (p+n\lambda_2)\, b_{p+n} \\ & \leq p - \alpha_2, \quad \text{since } g \in F_p^{c_2}(\lambda_1, \lambda_2).\end{aligned}$$

This shows that $f{*}g \in F_p^{c_1+c_2}(\lambda_2, \alpha_2)$.

Hence $f{*}g \in F_p^{c_1+c_2+1}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2}(\lambda_2, \alpha_2)$.

**Corollary.** *If* $f \in F_p^{c_1}(\lambda_1, \alpha_1)$ *and* $g \in F_p^{c_2}(\lambda_2, \alpha_2)$, *where* $0 \leq \lambda_1 < 1$ *and* $\lambda_2 \geq 1$, *then the quasi-Hadamard product* $f{*}g$ *belongs to* $F_p^{c_1+c_2}(\lambda_1, \alpha_1)$ *or* $F_p^{c_1+c_2}(\lambda_1, \alpha_2)$ *according as* $\alpha_1 \leq \alpha_2$ *or* $\alpha_1 \geq \alpha_2$.

**Proof**. We have

$$\begin{aligned} F_p^{c_1+c_2+1}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2}(\lambda_2, \alpha_2) & \\ & \subset F_p^{c_1+c_2}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2}(\lambda_2, \alpha_2) \\ & \subset F_p^{c_1+c_2}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2}(\lambda_1, \alpha_2) \\ & = F_p^{c_1+c_2}(\lambda_1, \alpha_1) \text{ or } F_p^{c_1+c_2}(\lambda_1, \alpha_2) \\ & \quad \text{according as } \alpha_1 \leq \alpha_2 \text{ or } \alpha_1 \geq \alpha_2. \end{aligned}$$

The result follows now from the above theorem.

## REFERENCES

[1] Subhas S. Bhoosnurmath and S.R. Swamy. *Modified Hadamard product of certain analytic functions*, Pure Appl. Math. Sci. **35**(1992), 71-77.

[2] Vinod Kumar. *Hadamard product of certain starlike fucntions*, J. Math. Anal. Appl. **110**(1985), 425-428.

[3] Vinod Kumar. *Quasi-Hadamard product of certain univalent functions*, J. Math. Anal. Appl. **126**(1987), 70-77.

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# ON A SPECIAL KAEHLERIAN RECURRENT SPACE OF SECOND ORDER

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**1. INTRODUCTION -** An $n(=2m)$ dimensional Kaehlerian space $K_n$ is a Riemannian space, which admits a structure field $F_i^h$, satisfying the relations (Yano 1965) :

$$F_i^h F_h^j = -\delta_i^j, \qquad \text{... (1.1)}$$

$$F_{ij} = -F_{ji}, (F_{ij} = F_i^a g_{aj}) \qquad \text{... (1.2)}$$

and $$\nabla_j F_i^h = 0, \qquad \text{... (1.3)}$$

where $\nabla$ denotes covariant differentiation with respect to the linear symmetric connection of the space.

The Riemannian curvature tensor, which we denote by $R_{ijk}^h$ is given by

$$R_{ijk}^h = \partial_i \left\{ \begin{matrix} h \\ jk \end{matrix} \right\} - \partial_j \left\{ \begin{matrix} h \\ ik \end{matrix} \right\} + \left\{ \begin{matrix} h \\ ia \end{matrix} \right\} \left\{ \begin{matrix} a \\ jk \end{matrix} \right\} - \left\{ \begin{matrix} h \\ ja \end{matrix} \right\} \left\{ \begin{matrix} a \\ ik \end{matrix} \right\}.$$

whereas the Ricci tensor and the scalar curvature are respectively given by $R_{ij} = R_{aij}^a$ and $R = R_{ij} g^{ij}$.

Let us, now, consider an affinely connected $n$-dimensional Kaehlerian recurrent space of second order, whose curvature tensor $R_{ijk}^h$ satisfies the following condition :

$$\nabla_n \nabla_m R_{ijk}^h = \lambda_{mn} R_{ijk}^h, \qquad \text{... (1.4)}$$

where $\lambda_{mn}$ is a non-symmetric, in general and non-vanishing covariant tensor. We shall assume to put the following two conditions in our space :

$$\nabla_j v^i = \phi_j v^i \qquad \text{... (1.5)}$$

and $$R_{jk} = \phi_j \alpha_k , \qquad \text{... (1.6)}$$

where $\alpha_k$ means a suitable covariant tensor, $R_{kl}$, we have

$$\nabla_n \nabla_m R_{ij} = \lambda_{mn} R_{ij} \qquad \text{... (1.7)}$$

Making use of (1.6) in (1.7), we have

$$\nabla_n \alpha_j \nabla_m \phi_i + \alpha_j \nabla_n \nabla_m \phi_i + \nabla_n \phi_i \nabla_m \alpha_j + \phi_i \nabla_n \nabla_m \alpha_j = \alpha_{mn} \phi_i \alpha_j \quad \text{... (1.8)}$$

In fact, when the space under consideration admits an affine motion

$$\bar{x}^i = x^i + v^i \delta t.$$

Characterized by (1.5), we have a resolved form of Ricci-tensor of the form (1.6). Here, we assume the existence of recurrent cavariant vector $v^i$ given by (1.5) and in addition, the resolvability of $R_{ij}$.

Making a commutator on the indices $m$ and $n$ in (1.8), we obtain

$$- \alpha_j \phi_a R^a_{imn} - \phi_i \alpha_a R^a_{jmn} = A_{mn} \phi_i \alpha_j \qquad \text{... (1.9)}$$

where $$A_{mn} \overset{\text{def}}{=\!=} \lambda_{mn} - \lambda_{nm}.$$

Multiplying (1.9) by $v^i$ and summing over $i$ and making use of (1.8), we have

$$\phi (A_{mn} \alpha_j + \alpha_a R^a_{jmn} + \alpha_j \Omega_{mn}) = 0,$$

where $$\phi \overset{\text{def}}{=\!=} \phi_i v^i.$$

In the present paper, we have to discuss the next two cases :

$$A_{mn} \alpha_j + \alpha_a R^a_{jmn} + \alpha_j \Omega_{mn} = 0 \qquad \text{... (1.10a)}$$

and $$\phi = 0. \qquad \text{... (1.10b)}$$

**2. THE CASE OF $\alpha = 0$.** Putting $\alpha = \alpha_m v^m$, the condition (1.10a) yields this case. We shall show this fact. Making use of (1.8) in (1.10a), we have

$$\alpha (A_{mn} + 2\Omega_{mn}) = 0.$$

Consequently, in the present case (1.10 a), we have to consider two cases. The first is a case of $\alpha = 0$ and the second is that of

$$A_{mn} + 2\Omega_{mn} = 0.$$

In the latter case, multiplying this condition by $v^h$ and making use of (1.8), we have

$$A_{mn} v^h + 2R^h_{amn} v^a = 0,$$

from which, by contraction on $h$ and $n$, it follows that

$$A_{mn} v^n + 2R_{am} v^a = 0,$$

or

$$A_{mn} v^n + 2\phi\alpha_m = 0,$$

i.e., the second case means the first case and (1.10 a) may be replaced by $\alpha = 0$. By this reason, there exists only one case of $\alpha \neq 0$.

From (1.4), we have

$$A_{mn} R^h_{ijk} = R^h_{amn} R^a_{ijk} - R^a_{imn} R^h_{ajk} + R^a_{jmn} R^h_{ika} - R^a_{kmn} R^h_{ija} \quad \text{... (2.1)}$$

Contracting the indices $h$ and $k$ in (2.1), we have

$$A_{mn} R_{ij} = - R^a_{imn} R_{aj} - R^a_{jmn} R_{ia} . \quad \text{... (2.2)}$$

Multiplying (2.2) by $v^i$ and summing on $i$, because of $v^i R_{ij} = v^i \phi_i \alpha_j = \phi\alpha_j$ and $\phi = 0$, we have

$$\alpha_a R^a_{jmn} = - (A_{mn} + \Omega_{mn})\alpha_j , \quad \text{... (2.3)}$$

where we have used (1.8) introducing (2.3) into the left hand side of (1.9), we have

$$\alpha_k (\phi_a R^a_{jmn} - \phi_j \Omega_{mn}) = 0.$$

Thus, there exist the following two cases to be discussed :

$$\alpha_k = 0 \quad \text{... (2.4a)}$$

$$\phi_a R^a_{jmn} = \phi_j \Omega_{mn} \quad \text{... (2.4b)}$$

In case of (2.4b), let us remember the first formula due to L.Bianchi, then we obtain

$$\phi_j \Omega_{mn} + \phi_m \Omega_{nj} + \phi_j \Omega_{jm} = 0. \quad \text{... (2.5)}$$

On the other hand, as we have

$$\Omega_{mn} = -\Omega_{nm} \text{ and } \phi = \phi_j v^j ,$$

we derive $\Omega_{jk} v^i = -R_{ak} v^a = -\phi \alpha_k$ .

from (1.8) by contraction on $h$ and $j$.

Multiplying (2.5) by $v^j$ and summing on $j$, we obtain

$$\Omega_{mn} = \alpha_m \phi_n - \alpha_n \phi_m . \qquad \text{... (2.6)}$$

Remembering the definition of $\Omega_{mn}$ and (1.6), we can write down an interesting formula :

$$R_{jk} - R_{kj} = \nabla_j \phi_k - \nabla_k \phi_j .$$

As we have (2.4b) and (1.8), we see that

$$\phi R^a_{jmn} v^h = \phi_j \Omega_{mn} v^h \text{ and } R^h_{amn} v^a = \Omega_{mn} v^h .$$

Hence, we have

$$\phi_a R^a_{jmn} v^h = \phi_j R^h_{amn} v^a$$

and this means that

$$R^a_{jmn} \nabla_a v^h - R^h_{amn} \nabla_j v^a = 0,$$

i.e., we have actually

$$\nabla_n \nabla_m (\nabla_j v^i) - \nabla_m \nabla_n (\nabla_j v^i) = 0.$$

Consequently, we can imagine the existence of a gradient vector $\rho_k$ and we are able to put

$$\nabla_k \nabla_j v^h = \rho_k \nabla_j v^h \qquad \text{... (2.7)}$$

From (1.5) and (2.7), we can deduce the equation :

$$\nabla_k \phi_j + \phi_j \phi_k = \rho_k \phi_j .$$

Multiplying the above equation by $v^j$, we have the following :

$$\begin{aligned} \phi \rho_k &= v^j \nabla_k \phi_j + \phi \phi_k \\ &= \nabla_k (v^j \phi_j) - \phi_j \nabla_k v^j + \phi \phi_k \\ &= \nabla_k \phi - \phi_j \phi_k v^j + \phi \phi_k \\ &= \nabla_k \phi - \phi \phi_k + \phi \phi_k = \nabla_k \phi . \end{aligned}$$

In this way, the existence of $\phi_j$ is examined and we have here a characteristic condition on $\nabla_j v^h$ :

$$\nabla_k \nabla_j v^h = \rho_k \nabla_j v^h, \rho_k = \frac{\nabla_k \phi}{\phi}.$$

On the other hand, in case of (2.4a), from (1.6), we obtain $R_{jk} = 0$.

Summerizing the above all conditions, we have the following :

**Theorem (2.1)** . In a $n$-dimensional Kaehlerian recurrence space of second order, admitting a contravariant vector $v^h$, characterized by

$$\nabla_j v^i = \phi_j v^i$$

and having a disjointed Ricci tensor of the form $R_{jk} = \phi_j \alpha_k$, there exist a case of $\alpha_k v^m = 0$.

In this case, if $\alpha_m = 0$, then we have the vanishing of Ricci tensor $R_{jk}$ and if $\alpha_m \neq 0$, we have

$$R_{kl} - R_{lk} = \nabla_l \phi_k - \nabla_k \phi_l.$$

The mixed tensor $\nabla_j v^i$ itself is a recurrent one characterized by

$$\nabla_l \nabla_k v^i = \rho_l \nabla_k v^i$$

for a definite gradient vector $\rho_l = \nabla_l \phi / \phi$.

**Remark .** The latter part of the above theorem does not contain a case of $\phi_j$ being a gradient vector.

Such a case is a showing but worthless one, in fact, if $\phi_j$ will be a gradient vector, we have

$$\alpha_l \phi_k - \alpha_k \phi_l = \nabla_k \phi_l - \nabla_l \phi_k = 0,$$

from which we have

$$\lambda \phi_k v^k = \alpha_k v^k = \alpha = 0,$$

this is $\lambda \phi = 0$ from which being $\phi \neq 0$, it follows that $\lambda = 0$.

Hence, we have $\alpha_k = 0$.

This a contradiction, for we have $\nabla_l \phi_k + \phi_k \phi_l = 0$ at this moment and the case of $\phi_k$ being a gradient vector occurs.

**3. The Case of $\phi = 0$.** Let us consider the case of (1.10b). Then using the analogous method used in §2, multiplying (2.2) by $v^j$, we obtain the following formula :

$$\phi_a R^a_{jmn} = -(A_{mn} + \Omega_{mn})\,\phi_j \qquad \text{... (2.8)}$$

Substituting (2.8) into the left hand side of (1.9), we get

$$\phi_j (\alpha_a R^a_{kmn} - \alpha_k \Omega_{mn}) = 0.$$

Hence, we have here two cases to be discussed. They are

$$\phi_j = 0 \qquad \text{... (2.9a)}$$

and
$$\alpha_a R^a_{kmn} = \alpha_k \Omega_{mn} \qquad \text{... (2.9b)}$$

The case of (2.9a) yields one of $\nabla_i v^h = 0$ and $R_{ij} = 0$.

The case of (2.9b) may be treated as follows :

We have

$$\alpha_j \Omega_{mn} + \alpha_m \Omega_{nj} + \alpha_n \Omega_{jm} = 0,$$

from which by contraction of $v^j$, we have

$$\alpha\Omega_{mn} = 0,$$

because of $\Omega_{jk} v^j = -R_{ak} v^a = -\phi_a \alpha_k v^a = -\phi\alpha_k$.

From (1.8), by contraction on the indices $h$ and $j$, and $\Omega_{mn} = -\Omega_{nm}$ and $\phi = \phi_j v^j$.

In this way, we obtain $\Omega_{mn} = 0$, say $\nabla_n \phi_m = \nabla_m \phi_n$.

Thus, we can state here the following :

**Theorem 2.2 .** When $\phi = 0$ in our space, there exist two cases. One of them is a case of satisfying $\nabla_j v^i = 0$ and $R_{jk} = 0$ and the other is a case of $\nabla_l \phi_k = \nabla_k \phi_i$ .

## REFERENCES

[1] K. Yano, *Differential Geometry on Complex and Almost Complex Spaces*, Pergamon Press, London (1965).

[2] K. Takano, On a space with bi-recurrent curvature, *Tensor (N.S.)*, **22** (1971), 329-332.

[3] K. Takano, and T. Imai, On some types of affine motions in bi- recurrent spaces, *Tensor (N.S.)*, **24** (1972), 93-100.

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# ON THE SUMMABILITY OF DOUBLE ORTHOGONAL SERIES

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Let $\phi_{ij}(x, y)$, $(i, j = 0, 1, 2, ...)$ be a sequence of orthonormal functions in the rectangle $R\ [a \le x \le b, c \le y \le d]$ i.e.

$$\int_R\int \phi_{ij}(x, y)\, \phi_{k,\, l}(x, y)\, dx\, dy = \begin{cases} 1 & \text{for } k = i,\ l = j \\ 0 & \text{for } k \ne i,\ l \ne j \end{cases}$$

Consider the double orthogonal series

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij}\, \phi_{ij}(x, y) \qquad \text{... (1.1)}$$

where $\{a_{ij}\}$ is an arbitrary sequence of real numbers.

The *mn*th partial sum of the series (1.1) is given by

$$s_{mn}(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij}\, \phi_{ij}(x, y).$$

The series (1.1) is said to be $(R, \lambda, \mu, 1, 1)$ summable to a function $s(x, y)$, if

$$\lim_{m,\, n \to} \sigma_{mn}(\lambda, \mu, x, y) = s(x, y)$$

where

$$\sigma_{mn}(\lambda, \mu, x, y) = \sum_{k=0}^{m} \sum_{l=0}^{n} \left(1 - \frac{\lambda_k}{\lambda_{m+1}}\right)\left(1 - \frac{\mu_l}{\mu_{n+1}}\right) a_{kl}\, \phi_{kl}(x, y)$$

The *mn*th Euler mean of the series (1.1) is given by

$$\tau_{mn}(x, y) = \frac{1}{2^{m+n}} \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k}\binom{n}{l} s_{kl}(x, y).$$

The series (1.1) is said to be $(E,1,1)$ summable to a function $s(x, y)$, if

$$\lim_{m, n \to \infty} \tau_{mn}(x, y) = s(x, y).$$

The convergence and Casáro summability of single orthogonal series have been studied by Alexits[2], Kaczmarz[3], Menchoff[6], Rademacher[10], Tandori[12] and Zygmund[13] and those of double orthogonal series by Agnew[1], Fedulov[4], Pandzakidze[8], Mitchell[7], Patel[9] and Sapre[11]. Dealing with (C,1,1) summability of double orthogonal series (1.1) Patel[9] has proved the following theorem :

**Theorem.** If

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl}^2 < \infty \qquad \dots (1.2)$$

Then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(\log m \log n)^p} (s_{mn}(x, y) - \sigma_{mn}(x, y))^2, p > 1$$

is convergent almost everywhere in $R$, where $\sigma_{mn}(x, y)$ is the $(C,1,1)$ mean of the series (1.1) We extend in this paper the above result to Riesz and Euler summability as follows :

**Theorem 1.** Under the condition (1.2)

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(\log m \log n)^p} [s_{mn}(x, y) - \sigma_{mn}(\lambda, \mu, x, y)]^2, p > 1$$

converges almost everywhere in $R$.

**Theorem 2.** Under the condition (1.2)

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(\log m \log n)^p} [s_{mn}(x, y) - \tau_{mn}(x, y)]^2, p > 1$$

converges almost everywhere in $R$.

In order to prove the above theorems, we need the following lemmas :

**Lemma 1 [5].** Writing

$$W_{nk} = \frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} - \frac{2k}{n+1},$$

we have

$$W_{nk} < 0 \left[ \frac{n}{3} \right] + 2 \leq k \leq n.$$

**Lemma 2 [5].**

$$\frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i}^2 \leq A \frac{k^2}{n^2} \text{ for } 1 \leq k \leq \left[ \frac{n}{3} \right] + 1$$

**2. Proof of Theorem 1.** We have

$$s_{mn}(x, y) - \sigma_{mn}(\lambda, \mu, x, y)$$

$$= \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl}\, \phi_{k\lambda}(x, y) - \sum_{k=0}^{m} \sum_{l=0}^{n} \left(1 - \frac{\lambda_k}{\lambda_{m+1}}\right)\left(1 - \frac{\mu_1}{\mu_{n+1}}\right) a_{kl}\, \phi_{kl}(x, y)$$

$$= \sum_{k=0}^{m} \sum_{l=0}^{n} \left( \frac{\lambda_k}{\lambda_{m+1}} + \frac{\mu_l}{\mu_{n+1}} - \frac{\lambda_k \mu_l}{\lambda_{m+1} \mu_{n+1}} \right) a_{kl}\, \phi_{kl}(x, y)$$

$$= \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\lambda_k}{\lambda_{n+1}} a_{kl}\, \phi_{kl}(x, y) + \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\mu_1}{\mu_{n+1}} a_{kl}\, \phi_{kl}(x, y)$$

$$- \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\lambda_k \mu_i}{\lambda_{m+1} \mu_{n+1}} a_{kl}\, \phi_{kl}(x, y).$$

Now, from the inequality

$$(\alpha_1 + \alpha_2 + \alpha_3)^2 \leq 3(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)$$

and the orthonormality property of the functions $\phi_{ij}(x, y)$, we get

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn (\log m \log n)^p} \int_R \int [s_{mn}(x, y) - \sigma_{mn}(\lambda, \mu, x, y)]^2 \, dx\, dy$$

$$\leq 3 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn\,(\log m \log n)^p} \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\lambda_k^2}{\lambda_{m+1}^2}\, a_{kl}^2 +$$

$$+ 3 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn\,(\log m \log n)^p} \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\mu_l^2}{\mu_{n+1}^2}\, a_{kl}^2$$

$$+ 3 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn\,(\log m \log n)^p} \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\lambda_k^2\,\mu_l^2}{\lambda_{m+1}^2\,\mu_{n+1}^2}\, a_{kl}^2$$

$= 3\,\Sigma_1 + 3\,\Sigma_2 + 3\,\Sigma_3$, say.

Now,

$$\sum\nolimits_1 = \sum_{m=1}^{\infty} \frac{1}{m\,(\log m)^p\,\lambda_{m+1}^2} \sum_{k=1}^{m} \lambda_k^2 \sum_{n=1}^{\infty} \frac{1}{n\,(\log n)^p} \sum_{l=1}^{n} a_{kl}^2$$

$$= \sum_{k=1}^{\infty} \lambda_k^2 \sum_{m=k}^{\infty} \frac{1}{m\,(\log m)^p\,\lambda_{m+1}^2} \sum_{l=1}^{\infty} a_{kl}^2 \sum_{n=l}^{\infty} \frac{1}{n\,(\log n)^p}$$

$$\leq \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} \frac{1}{m\,(\log m)^p} \sum_{l=1}^{\infty} a_{kl}^2 \sum_{n=l}^{\infty} \frac{1}{n\,(\log n)^p}$$

$$= O(1) \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl}^2 < \infty\,.$$

The convergence of $\Sigma_2$ and $\Sigma_3$ follows on the same line.

Hence by B.Levy's theorem, it follows that the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn\,(\log m \log n)^p}\, [s_{mn}(x, y) - \sigma_{mn}(\lambda, \mu, x, y)]^2$$

converges almost everywhere in $R$.

This completes the proof of the theorem.

**3. Proof of Theorem 2.** We have

$s_{mn}(x, y) - \tau_{mn}(x, y)$

$$2^{m+n} \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl}\,\phi_{kl}(x, y) \sum \sum \text{[illegible]}$$

$$-\frac{1}{2^{m+n}}\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\sum_{k=0}^{i}\sum_{l=0}^{j}a_{kl}\;\phi_{kl}(x,y)$$

$$=\frac{1}{2^{m+n}}\sum_{k=0}^{m}\sum_{l=0}^{n}a_{kl}\;\phi_{kl}(x,y)\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}$$

$$-\frac{1}{2^{m+n}}\sum_{k=0}^{m}\sum_{l=0}^{n}a_{kl}\;\phi_{kl}(x,y)\sum_{i=k}^{m}\sum_{j=l}^{n}\binom{m}{i}\binom{n}{j}$$

$$=\frac{1}{2^{m+n}}\sum_{k=0}^{m}\sum_{l=0}^{n}\left[\sum_{i=0}^{m}\binom{m}{i}\sum_{j=0}^{l-1}\binom{n}{j}+\sum_{i=0}^{k-1}\binom{m}{i}\sum_{j=l}^{n}\binom{n}{j}\right]a_{kl}\;\phi_{kl}(x,y)$$

$$=\frac{1}{2^{m+n}}\sum_{k=0}^{m}\sum_{l=0}^{n}2^{m}\sum_{j=0}^{l-1}\binom{n}{j}a_{kl}\;\phi_{kl}\;(x,y)$$

$$+\frac{1}{2^{m+n}}\sum_{k=0}^{m}\sum_{l=0}^{n}a_{kl}\;\phi_{kl}(x,y)\sum_{i=0}^{k-1}\binom{m}{i}\sum_{j=l}^{n}\binom{n}{j}.$$

Now, from the inequality

$$(\alpha_1+\alpha_2)^2\le 2\,(\alpha_1^2+\alpha_2^2)$$

and the orthonormality property of the functions $\{\phi_{ij} : (x, y)\}$, we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\,(\log m\,\log n)^p}\iint_R[s_{mn}(x,y)-\tau_{mn}(x,y)]^2\,dx\,dy$$

$$\le 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\,(\log m\,\log n)^p}\sum_{k=0}^{m}\sum_{l=0}^{n}a_{kl}^2\left\{\frac{1}{2^n}\sum_{j=0}^{l-1}\binom{n}{j}\right\}^2$$

$$+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\,(\log m\,\log n)^p}\sum_{k=0}^{m}\sum_{l=0}^{n}a_{kl}^2\left\{\frac{1}{2^{m+n}}\sum_{i=0}^{k-1}\binom{m}{i}\sum_{j=l}^{n}\binom{n}{j}\right\}^2$$

$$=O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\,(\log m\,\log n)^p}\sum_{k=1}^{m}\sum_{l=1}^{n}a_{kl}^2\,\frac{l^2}{n^2}$$

$$+O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\,(\log m\,\log n)^p}\sum_{k=1}^{m}\sum_{l=1}^{n}a_{kl}^2\,\frac{k^2}{m^2}$$

$$= O(1) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn\,(\log m \log n)^p} \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl}^2$$

$$= O(1) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m\,(\log m)^p} \sum_{l=1}^{\infty} a_{kl}^2 \sum_{n=l}^{\infty} \frac{1}{n\,(\log n)^p}$$

$$= O(1) \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl}^2 < \infty.$$

Hence, it follows by B.Levy's theorem that the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn\,(\log m \log n)^p} \left[s_{mn}(x, y) - \tau_{mn}(x, y)\right]^2$$

converges almost everywhere.

With this the theorem is proved.

## REFERENCES

[1] R.P. Angew, On double orthogonal series, *Proc. London Math. Soc.* **33**(2) (1931-32), 420-434.

[2] G. Alexits, Ein summationssatz für Orthogonalreihen *Acta.. Math. Acad. Sci. Hungar*. **7** (1959) 5-9.

[3] S. Kaczmarz, Uber die summierbarkeit der Orthogonalreihen *Math. Zeit.* **26**, (1927), 99-105.

[4] V.S. Fedulov, On (*C,1,1*) summability of double orthogonal series. *Ukrain Math. ZH* **7** (1955), 433-442.

[5] J. Meder, On the summability almost everywhere of orthogonal series by the method of Euler Knopp *Ann. Polon Math.* **5** (1958), 135-148.

[6] D. Menchoff, Sur les series de functions orthogonales I, *Fund. Math.* **4** (1923), 82-105.

[7] J. Mitchell, Convergence and (*C,1,1*) - summability of double orthogonal series. *Duke. Math. Journal* **18** (1951), 211-219.

[8] S.P. Pandzakidze, On the theorem of Menchov and Rademacher for double orthogonal series. *Soobsc Akad. Nauk. Gruzin SSR.* **39** (1965), 277-282.

[9] C.M. Patel, On double orthogonal series *Proc. Nat. Acad. Sci.* Sec. A. **37** Part II (1967), 241-244.

[10] H. Rademacher, Einige sätze Über Reihen von allgemeinen orthogonal funktionen *Math. Annalen* **87** (1922) 112-138.

[11] A.R. Sapre, On Euler means and *T*-means of orthogonal series *Mat. Vesnik.* **9** (1972) 91-96.

[12] K. Tandori, über die orthogonalen Funktionen II (summation) *Acta. Sci. Math.* **18** (1957), 149-168.

[13] A. Zygmund, Sur l'application de la premiére movenne arithmetique dans la theoree séries de fonctions orthogonales *Fund. Math.* **10** (1927), 355-362.

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# FUNCTIONS OF CLASS *Lip* $(\alpha, p)$ AND THEIR $F(a, q)$ MEAN

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## ABSTRACT

In the present paper some results of Mohapatra, Holland and Sahney [6] on Taylor mean are extended to general class of $F(a, q)$ transform.

## 1. INTRODUCTION

Let $f \in L[-\pi, \pi]$ and be periodic with period $2\pi$ outside this range. Let the Fourier series associated with $f(x)$ be given by

$$f(x) \sim \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \qquad \ldots (1.1)$$

We write

$$\varphi_x(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\} \qquad \ldots (1.2)$$

The family $F(a, q)$ of summability methods was introduced by Meir[5]. The summability matrix $\{c_{pk}\}$ belongs to $F(a, q)$ if it satisfies the following conditions :

$p$ is a discrete or continuous parameter; $q = q(p)$ is a positive increasing function which tends to infinity as $p \to \infty$, 'a' is a positive constant; for every fixed $\delta : \frac{1}{2} < \delta < \frac{2}{3}$.

$$c_p k = \sqrt{\frac{a}{\pi q}} \exp\left(-aq^{-1}(k-q)^2\right)\left\{1 + O\left(\frac{|k-q|+1}{q}\right) + O\left(\frac{|k-q|^3}{q^2}\right)\right\} \qquad \ldots (1.3)$$

as $p \to \infty$ uniformly in $k$ for $|k-q| \le q^{\delta}$; and

$$\sum_{|k-q| > q^{\delta}} (k+1)\, c_p\, k = O\{\exp(-q^{\mu})\} \qquad \ldots (1.4)$$

where μ is some positive number independent of $p$.

Let

$$l_p (f ; x) = \sum_{k=0}^{\infty} c_p k \, s_k (f ; x) \qquad \dots (1.5)$$

denote the $F(a, q)$ mean of the Fourier series (1.1) of $f$, where $s_k (f ; x)$ is the $k$th partial sum of (1.1).

The family $F(a, q)$ contains the summability methods of generalised Borel, Euler, Taylor, $S_\alpha$ and Valiron.

It is known (see Kuttner, Rajagopal and Rangachari[4]) that

$$\sum_{k=0}^{\infty} c_p k = 1 + O(q^{-1/2}) \qquad \dots (1.6)$$

The summability methods of Euler, Taylor, $S_\alpha$ and Borel satisfy (1.6) in the stronger form

$$\sum_{k=0}^{\infty} c_p k = 1 \qquad \dots (1.7)$$

Norms considered in this chapter are $Lp$ $(p \geq 1)$ norms and throughout the paper norms are taken with respect to variable $x$. The classes $Lip\ \alpha$, $Lip\ (\alpha, p)$ $(p \geq 1)$ are as defined by Hardy and Littlewood [1, p.612] and Zygmund [7], p. 42, 45. The class $Lip\ (\alpha, p)$ reduces to $Lip\ \alpha$ with $p = \infty$.

For $p \geq 1$, $\delta > 0$, let $\omega (\delta, g)$ and $\omega_p (\delta, g)$ denote the modulus of continuity and integral modulus of continuity respectively of an appropriate function g.

**2.** In 1985 Mohapatra, Holland and Sahney [6] studied order of approximation by Taylor mean $T_n^r (f ; x)$ of the Fourier series of $f(x)$ when $f(x)$ belongs to the class $Lip\ (\alpha, p)$ and proved the following theorems :

**THEOREM A.** If $f \in Lip\ (\alpha, p)$, $0 < \alpha \leq 1$, $p > 1$, then

$$\| T_n^r (f ; x) - f(x) \| = O(n^{-\alpha\beta}) \qquad \dots (2.1)$$

for $0 < \beta < 1/2$.

**THEOREM B.** If $f \in Lp$, $(p > 1)$, then for $0 < \beta < 1$

$$\|T_n^r(f;x) - f(x)\| = O(\omega_p(1/n;f)) + O\left(\int_{a_n}^{(a_n)^\beta} \frac{\omega_p(t;f)}{t}\,dt\right) + O(n^\beta \exp(-An^{1-2\beta})) \quad \ldots (2.2)$$

where $a_n = \pi\left(n + \frac{1}{2} + \frac{n+1}{1-r}r\right)^{-1}$.

**THEOREM C.** If $f \in Lip\ \alpha$, $0 < \alpha \leq 1$, $p > 1$, $\alpha p > 1$

$$T_n^r(f;x) - f(x) = O(n^{-\alpha - 1/p)\beta})\ (0 < \beta < 1/2) \quad \ldots (2.3)$$

uniformly in $x$ almost everywhere.

Since $F(a, q)$ method includes Taylor method, it is natural to ask as to what be the result if we apply $F(a, q)$ mean in place of Taylor mean in the above theorems. In answer to this question our aim in the present paper is to extend the above results to the family $F(a, q)$.

We prove the following results :

**THEOREM 1.** Let $[q]$ denote the integral part of $q = q(p)$ and $m = [q] + 1$. If $f \in Lp$, $(p > 1)$ and

$$\frac{\omega_p(t,f)}{t^{1/2}} \text{ is decreasing function of } t \text{ in } 0 \leq t \leq \pi \quad \ldots (2.4)$$

then, for $0 < \beta < 1$,

$$\|t_p(f;x) - f(x)\| = O(\omega_p^{(1/m,f)}) + O\left(\int_{\pi/m}^{\pi/m^\beta} \frac{\omega_p(t;f)}{t}\,dt\right) + O(m^\beta \exp(-Am^{1-2\beta})) \quad \ldots (2.5)$$

where $A = \frac{\pi^2}{4a}$, a constant.

**THEOREM 2.** If $f \in Lip\ (\alpha, p)$, $0 < \alpha \leq 1$, $p > 1$, then

$$\|t_p(f;x) - f(x)\| = O(m^{-\alpha\beta}) \quad \ldots (2.6)$$

for $0 < \beta < 1/2$, where $m$ is as in the Theorem 1.

**THEOREM 3.** If $f \in Lip\ (\alpha, p)$, $0 < \alpha \leq 1$, $p > 1$, $\alpha p > 1$

$$t_p(f;x) - f(x) = O(m^{-(\alpha - 1/p)\beta})\ (0 < \beta < 1/2) \quad \ldots (2.7)$$

uniformly in $x$ almost everywhere.

**3.** We shall need the following lemmas :

**LEMMA 1.** If $h(x, t)$ is a function of two variables defined for $0 \le t \le \pi$, $0 \le x \le 2\pi$, then

$$\| \int h(x, t)\, dt \| \le \int \| h(x, t)\, dt \|_p \, dt \ (p > 1).$$

This is due to Hardy, Littlewood and Polya[2].

**LEMMA 2.** If $q = q(p)$ is an integer valued function of $p$, then for $\frac{1}{2} < \delta < \frac{2}{3}$, we have

$$\int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-q| \le q^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-1)^2) \sin(k+1/2)t\, dt$$

$$= \int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \exp\left(-\frac{qt^2}{4a}\right) \sin(q+1/2)t\, dt + O(q \exp(-aq^{2\delta-1}))$$

**PROOF OF LEMMA 2.** From the proof of Lemma 3.2 of Ikeno[3], we have

$$\sum_{|k-q| \le q^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t$$

$$= \exp\left(-\frac{qt^2}{4a}\right) \sin(q+1/2)t\, dt + O(q \exp(-aq^{2\delta-1})\, |t|)$$

Hence the Lemma 2 follows in view of boundedness of $\varphi_x(t)$ and since $\sin t/2 > t/\pi$ $(0 < t < \pi)$.

**LEMMA 3.** If $m$ is as defined in the Theorem 1, we have

$$\int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-q| \le q^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t\, dt$$

$$= \int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-m| \le m^\delta} \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2) \sin(k+1/2)t\, dt + O(\omega_p(1/m, f))$$

**PROOF OF LEMMA 3.** We estimate the difference

$$\int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-q| \le q^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t\, dt$$

$$- \int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-m| \le m^\delta} \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2) \sin(k+1/2)t\, dt$$

$$= \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{m \le k \le m+m^{\delta}} \left[ \sqrt{\frac{a}{\pi q}} \exp\left(-aq^{-1}(k-q)^2\right) \right.$$

$$\left. - \sqrt{\frac{a}{\pi m}} \exp\left(-am^{-1}(k-m)^2\right) \right] \sin(k+1/2)\, t\, dt$$

$$+ \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{m-m^{\delta} \le k < m} \left[ \sqrt{\frac{a}{\pi q}} \exp\left(-aq^{-1}(k-q)^2\right) \right.$$

$$\left. - \sqrt{\frac{a}{\pi m}} \exp\left(-am^{-1}(k-m)^2\right) \right] \sin(k+1/2)t\, dt$$

$$- \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{q+q^{\delta} < k \le m+m} \sqrt{\frac{a}{\pi q}} \exp\left(-aq^{-1}(k-q)^2\right) \sin(k+1/2)t\, dt$$

$$+ \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{q-q^{\delta} \le k \le m-m^{\delta}} \sqrt{\frac{a}{\pi q}} \exp\left(-aq^{-1}(k-q)^2\right) \sin(k+1/2)t\, dt$$

$$= D_1 + D_2 + D_3 + D_4 \qquad \ldots (3.1)$$

From Ikeno[3], p.261, 262, we have

$$\sum_{q+q^{\delta} < k \le m+m^{\delta}} \sqrt{\frac{a}{\pi q}} \exp\left(-aq^{-1}(k-q)^2\right) \sin(k+1/2)t = O\left(\sqrt{q} \exp\left(-aq^{2\delta-1}\right)\right)$$

and also

$$\sum_{q-q^{\delta} < k \le m-m^{\delta}} \sqrt{\frac{a}{\pi q}} \exp\left(-aq^{-1}(k-q)^2\right) \sin(k+1/2)t = O\left(\sqrt{q} \exp\left(-aq^{2\delta-1}\right)\right)$$

Thus, using Lemma 1

$$\| D_3 \| \le \int_0^{\pi} \frac{\| \varphi_x(t) \|}{t} \sum_{q+q^{\delta} < k \le m+m^{\delta}} \sqrt{\frac{a}{\pi q}} \exp\left(-aq^{-1}(k-q)^2\right) |\sin(k+½)t|\, dt$$

$$= O\left(\sqrt{q} \exp\left(-aq^{2\delta-1}\right)\right). \qquad \ldots (3.2)$$

Similarly,

$$\| D_4 \| = O\left(\sqrt{q} \exp\left(-aq^{2\delta-1}\right)\right) \qquad \ldots (3.3)$$

In case where $q < m \le k \le m + m^{\delta}$, we have

$$O \le \frac{(k-m)}{\sqrt{m}} < \frac{(k-q)}{\sqrt{q}} < \frac{(k-[q])}{\sqrt{[q]}}\,.$$

Therefore, from Ikeno [3, p.250]

$$\begin{aligned}
&\left| \sqrt{\frac{a}{\pi q}} \exp\left(-aq^{-1}(k-q)^2\right) - \sqrt{\frac{a}{\pi m}} \exp\left(-am^{-1}(k-m)^2\right) \right| \\
&= O\left\{ \frac{1}{\sqrt{m}} \exp\left(-am^{-1}(k-m)^2\right) \left( \frac{(k-m)^2}{m^2} + \frac{|k-m|}{m} + \frac{1}{m} \right) \right\}
\end{aligned} \qquad \dots (3.4)$$

Using (3.3) and Lemma 1, we obtain

$$\begin{aligned}
\| D_1 \| &= O\left[ \int_0^{\pi} \frac{\| \varphi_x(t) \|}{t} \sum_{m \le k \le m + m^{\delta}} \left\{ \frac{1}{\sqrt{m}} \exp\left(-am^{-1}(k-m)^2\right) \right.\right. \\
&\qquad \left.\left. \left( \frac{(k-m)^2}{m^2} + \frac{|k-m|}{m} + \frac{1}{m} \right) \right\} \left| \sin(k+ \tfrac{1}{2})t \right| dt \right] \\
&= O\left[ \int_0^{\pi/m} \frac{\| \varphi_x(t) \|}{t} \sum_{m \le k \le m + m^{\delta}} \left\{ \frac{1}{\sqrt{m}} \exp\left(-am^{-1}(k-m)^2\right) \right.\right. \\
&\qquad \left.\left. \left( \frac{(k-m)^2}{m^2} + \frac{|k-m|}{m} + \frac{1}{m} \right) \right\} \left( |k-m| + m + \tfrac{1}{2} \right) t \, dt \right] \\
&\quad + O\left[ \int_{\pi/m}^{\pi} \frac{\| \varphi_x(t) \|}{t} \sum_{m \le k \le m + m^{\delta}} \left\{ \frac{1}{\sqrt{m}} \exp\left(-am^{-1}(k-m)^2\right) \right.\right. \\
&\qquad \left.\left. \left( \frac{(k-m)^2}{m^2} + \frac{|k-m|}{m} + \frac{1}{m} \right) \right\} dt \right] \\
&= O\left\{ \sqrt{m} \int_0^{\pi/m} \| \varphi_x(t) \| \, dt \right\} + O\left\{ \frac{1}{\sqrt{m}} \int_{\pi/m}^{\pi} \frac{\| \varphi_x(t) \|}{t} \, dt \right\} \\
&= O\left\{ \sqrt{m} \int_0^{\pi/m} \omega_p(t, f) \, dt \right\} + O\left\{ \frac{1}{\sqrt{m}} \int_{\pi/m}^{\pi} \frac{\omega_p(t, f)}{t} \, dt \right\} \\
&= O\left( \omega_p(1/m . f) \right)
\end{aligned} \qquad \dots (3.5)$$

Also in case where $k \le [q] \le q < m$, we get

$$\frac{(k-m)}{\sqrt{m}} < \frac{(k-q)}{\sqrt{q}} < \frac{(k-[q])}{\sqrt{[q]}} \le 0.$$

Hence from Ikeno [3, p.260] the following estimate results :

$$\left| \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) - \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2) \right|$$

$$= O\left( \frac{1}{\sqrt{[q]}} \exp(-a[q]^{-1}(k-[q])^2 \left( \frac{(k-[q])^2}{[q]^2} + \frac{|k-[q]|}{[q]} + \frac{1}{[q]} \right) \right)$$

Proceeding as in the estimation of $D_1$ and using the above inequality we get

$$\| D_2 \| = \omega_p(1/m, f). \qquad \text{... (3.6)}$$

Thus, Lemma 3 follows from (3.1) to (3.6).

**LEMMA 4.** Suppose that $f \in Lip(\alpha, p)$, where $p > 1$, $0 < \alpha \leq 1$, $\alpha p > 1$. Then $f$ is equal to a function $g \in Lip(\alpha - 1/p)$ almost everywhere.

This is due to Hardy and Littlewood[1], Theorem 5(ii), p.627.

**4. PROOF OF THEOREM 1.** Since

$$s_k(f;x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sin(k+1/2)t\, dt$$

we have

$$t_p(f;x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{k=0}^{\infty} c_p\, k \sin(k+1/2)t\, dt + O(q^{-1/2})$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \left[ \left( \sum_{|k-q| \leq q^{\delta}} + \sum_{|k-q| > q^{\delta}} \right) c_p\, k \sin(k+1/2)t \right] dt + O(q^{-1/2})$$

$$= S_1 + S_2 + O(q^{-1/2}). \qquad \text{... (4.1)}$$

By the Minkowski Inequality

$$\| t_p(f;x) - f(x) \| \leq \| S_1 \| + \| S_2 \| + O(q^{-1/2}) \qquad \text{... (4.2)}$$

By (1.4), Lemma 1 and the fact that $\sin t/2 \geq t/\pi$ $(O < t \leq \pi)$, we obtain

$$\| S_2 \| \leq \int_0^{\pi} \frac{\| \varphi_x(t) \|}{t} \sum_{|k-q| > q^{\delta}} c_p\, k\, (k+1/2)t\, dt = O(\exp(-q^{\mu})).$$

... (4.3)

Now making use of (1.3), we can write

$$S_1 = \frac{1}{\pi}\int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-q| \le q^\delta} \sqrt{\frac{a}{\pi q}}\ \exp(-aq^{-1}(k-q)^2)$$

$$\left\{1 + O\left(\frac{|k-q|+1}{q}\right) + O\left(\frac{|k-q|^3}{q^2}\right)\right\} \sin(k+1/2)t\,dt$$

$$= S_3 + S_4 + S_5. \qquad \text{... (4.4)}$$

Using Lemma 1, we estimate $S_4$ as follows :

$$\| S_4 \| \le \int_0^\pi \frac{\| \varphi_x(t) \|}{t} \sum_{|k-q| \le q^\delta} \left\{\sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)\, O\left(\frac{|k-q|+1}{q}\right)\right\}$$

$$| \sin(k + 1/2)t \mid dt$$

$$= \int_0^{\pi/q} \frac{\| \varphi_x(t) \|}{t} \sum_{|k-q| \le q^\delta} \left\{\sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)\, O\left(\frac{|k-q|+1}{q}\right)\right\}$$

$$(|k-q| + q + 1/2)t\,dt$$

$$+ \int_{\pi/q}^\pi \frac{\| \varphi_x(t) \|}{t} \sum_{|k-q| \le q^\delta} \sqrt{\frac{a}{\pi q}}\ \exp\ (-aq^{-1}(k-q)^2)\, O\left(\frac{|k-q|+1}{q}\right) dt$$

$$= O\left(\sqrt{q}\int_0^{\pi/q} \| \varphi_x(t) \| \, dt\right) + O\left(\frac{1}{\sqrt{q}}\int_{\pi/q}^\pi \frac{\| \varphi_x(t) \|}{t}\,dt\right)$$

$$= IO\,(\omega_p\,(1/q, f)). \qquad \text{... (4.5)}$$

Similarly,

$$\| S_5 \| = O(\omega_p\,(1/q, f)). \qquad \text{... (4.6)}$$

Applying Lemma 2 and Lemma 3 and noting that $m = m(p)$ is an integer valued function of $p$ we get

$$S_3 = \int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \exp\left(-\frac{mt^2}{4a}\right) \sin(m + 1/2)t\,dt + O\,(\omega_p\,(1/m, f))$$

$$\text{... (4.7)}$$

.Now we write

$$I_1 = \left(\int_0^{\pi/m} + \int_{\pi/m}^{\pi/m\,\beta} + \int_{\pi/m\,\beta}^\pi\right) \frac{\varphi_x(t)}{\sin t/2} \exp\left(-\frac{mt^2}{4a}\right) \sin(m + 1/2)t\,dt$$

$$= I_{1.2} + I_{1.2} + I_{1.3} \qquad \text{... (4.8)}$$

Since $\sin t/2 \le t/\pi$ $(0 < t \le \pi)$, we have by Lemma 1

$$\| I_{1.1} \| \le \pi \int_0^{\pi/m} \frac{\| \varphi_x(t) \|}{t} \exp\left( - \frac{mt^2}{4a} \right) \sin\,(m + 1/2)t\, dt$$

$$\le O\left( \int_0^{\pi/m} \frac{\omega_p(t, f)}{t} (m + 1/2)t\, dt \right)$$

$$= O(\omega_p(1/m, f)). \qquad \text{... (4.9)}$$

Further, by Lemma 1

$$\| I_{1.3} \| \le O\left( \int_{\pi/m\ \beta}^{\pi} \frac{\omega_p(t, f)}{t} \exp\left( - \frac{mt^2}{4a} \right) dt \right)$$

$$= O\,\{m^{\beta} \exp\,(- Am^{1 - 2\beta})\}. \qquad \text{... (4.10)}$$

Finally, by using Lemma 1 and the fact that $|\sin x| \le 1$ for all $x$, we obtain

$$\| I_{1.2} \| \le \int_{\pi/m}^{\pi/m\ \beta} \frac{\| \varphi_x(t) \|}{t} \exp\left( - \frac{mt^2}{4a} \right) dt$$

$$= \int_{\pi/m}^{\pi/m\ \beta} \frac{\omega_p(t, f)}{t} \exp\left( - \frac{mt^2}{4a} \right) dt$$

$$= O\left( \int_{\pi/m}^{\pi/m\ \beta} \frac{\omega_p(t, f)}{t}\, dt \right). \qquad \text{... (4.11)}$$

Estimates (4.1), (4.2), ..., (4.11) togther yield the proof of the Theorem 1.

**PROOF OF THE THEOREM 2.** For $\beta < \frac{1}{2}$, we have

$$m^{\beta} \exp\,(- Am^{1 - 2\beta}) = \omega_p(1/m, f). \qquad \text{... (4.12)}$$

Since $f \in Lip\,(\alpha, p)$, $0 < \alpha \le 1$, $p > 1$,

$$\omega_p(1/m, f) = O(m^{-\alpha}). \qquad \text{... (4.13)}$$

Now to deduce the Theorem 2 from Theorem 1 it suffices to show that

$$\int_{\pi/m}^{\pi/m\ \beta} \frac{\omega_p(t, f)}{t}\, dt = O\,(m^{-\alpha\beta}). \qquad \text{... (4.14)}$$

Above equation follows from the fact that

$$\int_{\pi/m}^{\pi/m} t^{\alpha - 1}\, dt = O\,(m^{-\alpha\beta}) + O(m^{-\alpha}).$$

**PROOF OF THE THEOREM 3.** Since $f \in Lip\,(\alpha, p)$, $0 \le \alpha < 1$, $p > 1$, $\alpha p > 1$ from Lemma 4 we see that there exists a function $g \in Lip\,(\alpha - 1/p)$, which is equal to $f$ almost everywhere. Hence, we have almost everywhere

$$\varphi_x(t) = O\,(t^{\alpha - 1/p}). \qquad \ldots (4.15)$$

Now in order to prove the Theorem 3 we modify estimates of $D_1, D_2, D_3, D_4$ in the proof of Lemma 3.

Using arguments similar to those used in estimating $D_3$

$$D_3 = O\left[\int_0^{\pi} \frac{|\varphi_x(t)|}{t} \sum_{q+q^{\delta} < k \le m+m^{\delta}} \sqrt{\frac{a}{\pi q}}\, \exp\,(-aq^{-1}(k-q)^2)\; |\sin\,(k + 1/2)t\;|\; dt\right]$$

$$= O\,(\sqrt{q}\, \exp\,(-aq^{2\delta-1})). \qquad \ldots (4.16)$$

Similarly,

$$D_4 = O\,(\sqrt{q}\, \exp\,(-aq^{2\delta-1})). \qquad \ldots (4.17)$$

As in the proof of Lemma 3, we get

$$D_1 = O\left\{\sqrt{m}\int_0^{\pi/m} |\varphi_x(t)|\; dt\right\} + O\left\{\frac{1}{\sqrt{m}}\int_{\pi/m}^{\pi} \frac{|\varphi_x(t)|}{t}\, dt\right\}$$

$$= O(m^{-\alpha + 1/p}). \qquad \ldots (4.18)$$

In a similar manner, we can obtain

$$D_2 = O(m^{-\alpha + 1/p}). \qquad \ldots (4.19)$$

Thus, combining (3.1), (4.16), (4.17) and (4.19), we have

$$\int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-q| \le q^{\delta}} \sqrt{\frac{a}{\pi q}}\, \exp\,(-aq^{-1}(k-q)^2)\, \sin\,(k + 1/2)t\; dt$$

$$= \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-m| \le m^{\delta}} \sqrt{\frac{a}{\pi m}}\, \exp\,(-am^{-1}(k-m)^2)\, \sin\,(k + 1/2)t\; dt$$

$$+\, O\,(m^{-\alpha + 1/p}) \qquad \ldots (4.20)$$

In the notations of the Theorem 1.

$$t_p\,(f\,;x) - f(x) \le S_1 + S_2 + O(q^{-1/2}). \qquad \ldots (4.21)$$

Using (1.4), we get

$$S_2 = O\left( \int_0^\pi \frac{|\varphi_x(t)|}{t} \sum_{|k-q|>q^\delta} c_p\, k\,(k+½)t\, dt \right)$$

$$= O(\exp(-q^\mu)). \qquad \text{... (4.22)}$$

From (4.4), we have

$$S_1 = S_3 + S_4 + S_5. \qquad \text{... (4.23)}$$

On the lines of the esimation of $S_4$ in the proof of the Theorem 1 and using (4.15), we obtain

$$S_4 = O\left( \sqrt{q} \int_0^{\pi/q} |\varphi_x(t)|\, dt \right) + O\left( \frac{1}{\sqrt{q}} \int_{\pi/q}^{\pi} \frac{|\varphi_x(t)|}{t}\, dt \right)$$

$$= O(q^{-\alpha + 1/p}). \qquad \text{... (4.24)}$$

Similarly,

$$S_5 = O(q^{-\alpha + 1/p}). \qquad \text{... (4.25)}$$

Applying Lemma 2 and (4.20) and using the fact that $m = m(p)$ is an integer valued function of $p$, we get

$$S_3 = \int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \exp\left( -\frac{mt^2}{4a} \right) \sin(m+½)t\, dt + O(m^{-\alpha + 1/p})$$

$$= I_1 + O(m^{-(\alpha + 1/p)}).$$

As in the Theorem 1, we write

$$I_1 = \left( \int_0^{\pi/m} + \int_{\pi/m}^{\pi/m\,\beta} + \int_{\pi/m\,\beta}^{\pi} \right) \frac{\varphi_x(t)}{\sin t/2} \exp\left( -\frac{mt^2}{4a} \right) \sin(m+½)t\, dt$$

$$= I_{1.1} + I_{1.2} + I_{1.3}\,. \qquad \text{... (4.26)}$$

Now using (4.15) we shall modify the estimates of $I_{1.1}$, $I_{1.2}$, $I_{1.3}$.

We have

$$I_{1.1} = O\left( \int_0^{\pi/m} \frac{|\varphi_x(t)|}{t} \exp\left( -\frac{mt^2}{4a} \right) \sin m + ½)t\, dt \right)$$

$$= O\left( \int_0^{\pi/m} \frac{t^{\alpha - 1/p}}{t} (m+½)t\, dt \right)$$

$$= O(m^{-(\alpha + 1/p)}). \qquad \text{... (4.27)}$$

Again,

$$I_{1.3} = O\left(\int_{\pi/m\,\beta}^{\pi} t^{\alpha - 1/p - 1} \exp\left(-\frac{mt^2}{4a}\right)dt\right)$$
$$= O\,(m^{-(\alpha - 1/p)\,\beta}). \qquad \ldots (4.28)$$

Lastly,

$$I_{1.2} = O\left(\int_{\pi/m}^{\pi/m\,\beta} \frac{|\varphi_x(t)|}{t} \exp\left(-\frac{mt^2}{4a}\right)dt\right)$$
$$= O\left(\int_{\pi/m}^{\pi/m\,\beta} t^{\alpha - 1/p - 1}\,dt\right)$$
$$= O\,(m - (\alpha - 1/p)\beta) + O\,(m^{-\alpha + 1/p})$$
$$= O\,(m^{-(\alpha - 1/p)\,\beta}). \qquad \ldots (4.29)$$

Hence the Theorem 3 follows from (4.21) to (4.29).

## REFERENCES

[1] Hardy, G.H. and Littlewood, J.E. : A convergence criterion for Fourier series, Math. Z **28** (1928), 612-634.

[2] Hardy, G.H., Littlewood, J.E. and Polya, G. : "Inequalities" Cambridge Univ. Press, London/New York (1967).

[3] Ikeno, K. : Lebesgue constants for family of summability methods, Tōhoku Math. J. **17** (1965), 250-265.

[4] Kuttner, B., Rajagopal, C.T. and Rangachari, M.S. : Tauberian convergence theorems for summability methods of Karamata family, J. Indian Math. Soc. (N.S.), **44** (1980), 23-38.

[5] Meir, A. : Tauberian constants for a family of transformations, Annals of Math. **78** (1963), 594-599.

[6] Mohapatra, R.N., Holland, A.S.B. and Sahney, B.N. : Functions of class *Lip* $(\alpha, p)$ and their Taylor mean, Journal of Approximation Theory **45** (1985), 363-374.

[7] Zygmund, A. : "Trignometric Series", Vols. I and II 2nd ed., Cambridge Univ. Press, London/New York (1968).

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# *MHD* FLUCTUATING HELE-SHAW FLOW OF A VISCOUS FLUID PAST A CIRCULAR CYLINDER

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## ABSTRACT

In the present problem an attempt has been made to study the effects of boundary fluctuations and the transverse magnetic field on the Hele-Shaw flow of incompressible viscous fluid past a circular cylinder $x^2 + y^2 = r^2, -d \leq z \leq d$.

**1. INTRODUCTION.** In 1897, Prof. Hele-Shaw established that the Hele-Shaw cell is an arrangement of two parallel plates very close together, the space between the plates being occupied partly by fluid and partly by obstacles in the form of cylinders with generators normal to the plates. Several researchers have paid their attention towards the Hele-Shaw flows. Lamb (1932), Thompson (1968), Lee & Fung (1969) and Buckmaster (1970) have discussed such flows. In these investigations, the pressure gradient was assumed to be a constant and the flow in the cell to be steady. Swaminathan (1975) studied the unsteady Hele-Shaw flow of viscous incompressible fluid under time dependent pressure gradient. Lal & Singh (1981) have extended the work of Swaminathan to discuss the unsteady Hele-Shaw flow through porous media under time varying pressure gradient.

Reigels (1989) has discussed *MHD* Hele-Shaw flow of Rivlin-Ericksen fluid. Sengupta and Roy (1991) have also investigated *MHD* flow of *n*-immiscible viscoelastic fluid under gravity between two parallel plates. Recently, Ghosh & Sengupta (1995) have studied *MHD* Hele-Shaw flow of Rivlin-Ericksen fluid.

In this paper, we have studied the effects of boundary fluctuations under the influence of transverse magnetic field on the Hele-Shaw flow under constant pressure. The expressions for extact velocity distributions have been found and graphically represented under the influence of magnetic field of different intensities.

**2. GOVERNING EQUATIONS OF MOTION.** Let the viscous incompressible fluid flows past a vertical cylinder $x^2 + y^2 = r^2$, $-d \le z \le d$; placed between two parallel plates $z = \pm d$.

The equations of motion for a viscous incompressible fluid in the unsteady Hele-Shaw cell under transverse magnetic field are

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u_2}{\partial z^2} - \frac{\sigma}{\rho} B_0^2 u \qquad \text{... (2.1)}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial z^2} - \frac{\sigma}{\rho} B_0^2 v \qquad \text{... (2.2)}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \text{... (2.3)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{... (2.4)}$$

where $\rho$ the density, $p$ the pressure of the field, $\sigma$ the electrical conductivity, $B_0$ the strength of magnetic field. The boundary conditions are

$$u = u_0\, e^{int},\ v = u_0\, e^{int} \text{ at } z = \pm d \qquad \text{... (2.5)}$$

Normal velocity at the surface of the cylinder is zero. ... (2.6)

$U$ is the finite and $V = 0$ as $|x|,\ |y| \to \infty$ ... (2.7)

**3. MATHEMATICAL ANALYSIS.** We introduce the following non-dimensional quantities

$$y' = \frac{y}{d},\ x' = \frac{x}{d},\ r' = \frac{r}{d},\ u' = \frac{u}{u_0},\ t' = \frac{\mu t}{\rho d^2},\ \nu' = \frac{\nu}{u_0},\ p' = \frac{pd}{\mu u_0},$$

$$d' = \frac{n\rho d^2}{\mu} \text{ and } M = B_0 d\sqrt{\sigma/\mu},$$

where $M$ is the Hartmann number.

The equations (2.1) to (2.4) under the constant pressure (after dropping dashes) reduce to :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial z^2} - M^2 u \qquad \text{... (3.0)}$$

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial z^2} - M^2 v \qquad \text{... (3.1)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{... (3.2)}$$

The boundary conditions, accordingly, are

$$u = v = e^{int} \text{ at } z = \pm 1 \qquad \dots (3.3)$$

The conditions for $\phi$ are

$$\frac{\partial \phi}{\partial r} = 0 \text{ at the surface of the cylinder where } x^2 + y^2 = R^2 \qquad \dots (3.4)$$

$$\frac{\partial \phi}{\partial x} = 1 \text{ and } \frac{\partial \phi}{\partial y} = 0 \text{ as } |x| \to \infty,\ |y| \to \infty \qquad \dots (3.5)$$

**4. SOLUTION.** Let

$$u = U(z)\, \frac{\partial \phi}{\partial x}\, e^{int},\ -1 \le z \le 1 \qquad \dots (4.1)$$

$$v = U(z)\, \frac{\partial \phi}{\partial y}\, e^{int},\ -1 \le z \le 1 \qquad \dots (4.2)$$

$$u = v = U(z)\, e^{int} \text{ at } z = \pm 1 \qquad \dots (4.3)$$

The conditions (3.3) suggests that

$$U(z) = 1 \text{ at } z = \pm 1 \qquad \dots (4.4)$$

Now substituting from (4.1) to (4.3) in equations (3.0) to (3.2), we get

$$\frac{\partial^2 u}{\partial z^2} - (M^2 + in)\, U = 0 \qquad \dots (4.5)$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \dots (4.6)$$

The solution of equation (4.5) under the conditions (4.4) can easily be obtained as

$$U = \frac{\cosh\,(\sqrt{M^2 + in})z}{\cosh\,(\sqrt{M^2 + in})} \qquad \dots (4.7)$$

Also the solution of (4.6) under the conditions (3.4) and (3.5) can be obtained as

$$\phi = \left( R + \frac{r^2}{R} \right) \cos \theta \qquad \dots (4.8)$$

Substituting from equations (4.7) and (4.8) in equations (4.1) and (4.2), the exact velocity distributions are obtained as

$$u = \frac{\cosh\,(\sqrt{M^2 + in})z}{\cosh\,(\sqrt{M^2 + in})} \left\{ 1 - \frac{r^2\,(x^2 - y^2)}{(x^2 + y^2)^2} \right\} e^{int} \qquad \dots (4.9)$$

and

$$v = \frac{\cosh\,(\sqrt{(M^2 + in})z}{\cosh\,(\sqrt{M^2 + in})} \left\{ - \frac{2r^2 xy}{(x^2 + y^2)^2} \right\} e^{int} \qquad \dots (4.10)$$

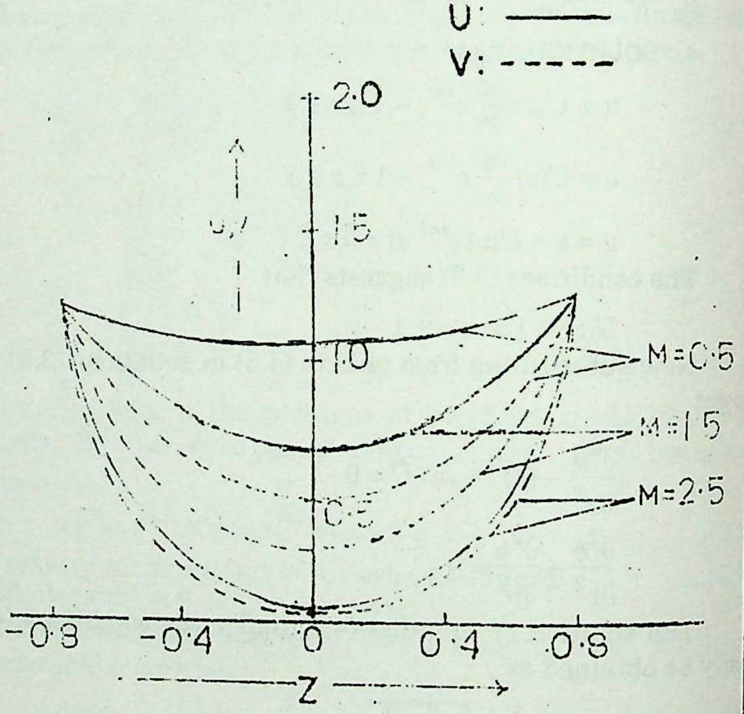

FIG.(I) VELOCITIES PLOTTED AGAINST Z
AT $n=1$, $ni=0$, $X=1$ AND $Y=0·1$

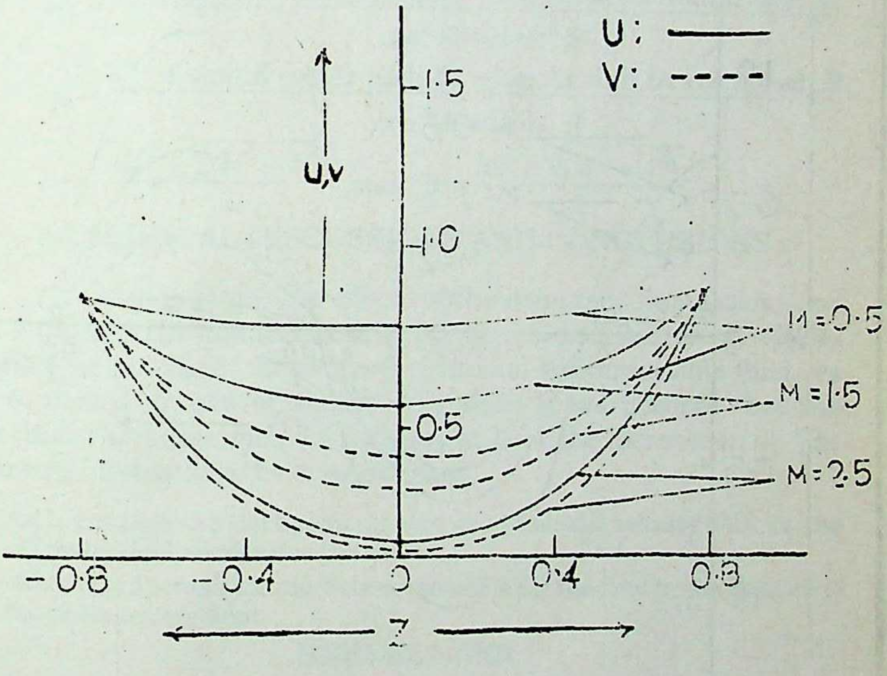

FIG.II VELOCITIES PLOTTED AGAINST Z AT $nt = \pi/4$, $n = 1$, $X = 1$, AND $Y = -1$

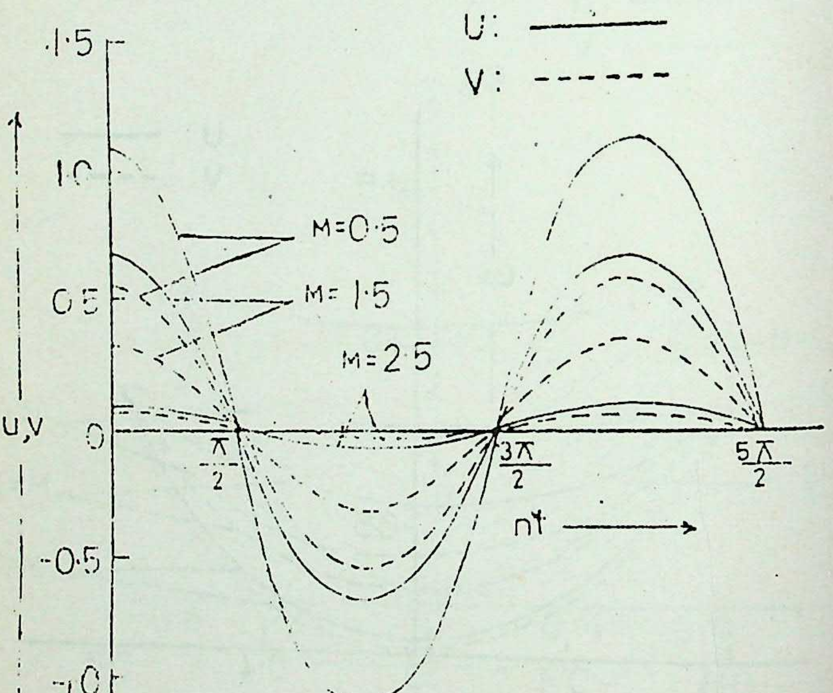

FIG. III VELOCITIES PLOTTED AGAINST nt AT Z=0, n=1, X=1 AND Y=−1

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Equating the real parts, we have

$$u = (M_r \cos nt - M_i \sin at)\left\{1 - \frac{r^2(x^2 - y^2)}{(x^2 + y^2)^2}\right\} \qquad \dots (4.11)$$

and

$$v = (M_r \cos nt - M_i \sin nt)\left\{-\frac{2r^2(x^2 - y^2)}{(x^2 + y^2)^2}\right\} \qquad \dots (4.12)$$

where

$$M_r = \frac{\cos A \cos Az \cosh B \cosh Bz + \sin A \sin Az \sinh B \sinh Bz}{\cos^2 A + \sinh^2 B}$$

$$M_i = \frac{\sin A \sinh B \cos Az \cosh Bz - \sin Az \sinh Bz \cos A \cosh B}{\cos^2 A + \sinh^2 B}$$

$$A = -\frac{M^2 + \sqrt{(M^4 + n^2)}}{2} \text{ and } B = \frac{M^2 + \sqrt{(M^4 + n^2)}}{2}.$$

## NUMERICAL DISCUSSIONS AND CONCLUSIONS :

In order to study the effects of the boundary fluctuations and the effects of Hartmann number (M) on the distribution of velocity profiles on the 'Hele - Shaw flow' of viscous incompressible fluid, we have plotted graphs of velocities against z, we observe that the velocities decreases with the increase in M or the increase in *nt*. The following results have been established.

[i] An Increase in the Hartmann number decreases the velocity field, i.e. the magnetic field decelerates the flow.

[ii] Boundary fluctuations are held responsible for the flow in the absence of the pressure gradient.

## REFERENCES

[1] H. Lamb, *Hydrodynamics*, 6th ed.(1932), Art 330.

[2] H.S. Carslaw and J.G. Jaegar, *Conduction of Heat in Solids*, Oxford/London (1959), 302-304.

[3] B.W. Thompson, Secondary Flow in a Hele-Shaw Cell, *Journal of Fluid Mechanics* (1968), 379.

[4] J.S. Lee and Y.G. Fung, *Jour. Fluid Mechanics*, **37**, (1969), 657.

[5] K. Swaminathan : Ph.D. Thesis, Agra University, Agra, 1975.

[6] G. Lal and D.P. Singh, Unsteady Hele-Shaw flow of viscous fluid through poroous medium. Presented at seminar organised by Department of Mathematics, B.H.U. Varaniasi (Sept. 1981)

[7] F. Reigels, *ZAMM*, **18**, (1989) p.89.

[8] P.R. Sen Gupta and T.K. Roy, *Ind. Jour. Theor. Phys.*, **39**, No.1 (1991).

[9] B.C. Ghosh and P.R. Sen Gupta, *Ind, Jour. Theor, Phys.*, **43** No. 2 (1995).

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